

# Reservoir Stack Machines

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## Abstract

Memory-augmented neural networks equip a recurrent neural network with an explicit memory to support tasks that require information storage without interference over long times. A key motivation for such research is to perform classic computation tasks, such as parsing. However, memory-augmented neural networks are notoriously hard to train, requiring many backpropagation epochs and a lot of data. In this paper, we introduce the reservoir stack machine, a model which can provably recognize all deterministic context-free languages and circumvents the training problem by training only the output layer of a recurrent net and employing auxiliary information during training about the desired interaction with a stack. In our experiments, we validate the reservoir stack machine against deep and shallow networks from the literature on three benchmark tasks for Neural Turing machines and six deterministic context-free languages. Our results show that the reservoir stack machine achieves zero error, even on test sequences longer than the training data, requiring only a few seconds of training time and 100 training sequences.

## 1 Introduction

Memory-augmented neural networks (MANNs) are models that extend recurrent neural networks with an explicit external memory (Graves et al., 2016; Rae et al., 2016; Santoro et al., 2016). A main motivation for such extensions is to enable computation tasks that require a separation of memory and computation, i.e. the network needs to keep track of information over long stretches of time without this information getting corrupted in every step of the recurrent update process (Graves et al., 2016). MANNs have achieved impressive successes on that front, performing computation tasks that were previously out of reach for neural networks, such as associative memory recalls, question-answering, and graph traversal (Csordás and Schmidhuber, 2019; Giles et al., 1989; Graves et al., 2016; Rae et al., 2016; Suzgun et al., 2019). Due to their ability to perform computation tasks, MANNs have also been named Neural Turing Machines or Differentiable Neural Computers (Graves et al., 2016).

While the successes of this line of research are impressive, MANNs are typically hard to train, requiring many epochs of gradient descent and a lot of training data (Collier and Beel, 2018). This is because one needs to backpropagate through a series of quasi-discrete read and write operations on the memory with difficult dependencies. For example: accessing the memory is only useful if the memory contains information that one needs to produce a certain

output, and the memory only contains such information if the network has previously put it into the memory.

In this paper, we introduce the *reservoir stack machine* (RSM), an echo state network (Jaeger and Haas, 2004) combined with an explicit stack that can store information without interference but is much simpler to train compared to the aforementioned MANN models thanks to two tricks. First, we follow the reservoir computing paradigm by leaving recurrent matrices fixed after initialization, limiting training to the output layers (Jaeger and Haas, 2004; Rodan and Tiño, 2012). Second, we do not require our network to learn optimal write/read behavior autonomously but provide training data for it. Note that this is a restriction of our model because we need more annotations per training sequence compared to standard recurrent neural networks. However, this additional labeling makes training vastly more efficient: Our training reduces to a classification problem that can be solved with convex optimization in seconds instead of minutes to hours, using only  $\approx 100$  short example inputs instead of hundreds of thousands. One could see this approach as an instance of imitation learning, where learning a complicated recurrent process becomes significantly simpler because we imitate the actions of a teacher on a few example demonstrations (Ross and Bagnell, 2010).

The RSM architecture is inspired by three main sources. First, we build on previous work regarding differentiable stacks (Giles et al., 1989; Mali et al., 2020; Suzgun et al., 2019) which suggests a stack as memory for neural networks to recognize typical context-free languages. Second, we build on classic parser theory (Knuth, 1965) to show that the RSM is at least as powerful as an LR(1)-automata, i.e. it can recognize all deterministic context-free languages. Finally, our model builds upon prior work in reservoir computing, which shows that echo state networks on their own have little computational power (below Chomsky-3; Hammer and Tiño, 2003) but are well suited to distinguish input sequences by a constant-sized suffix - which can be used to guide memory access behavior (Paaßen et al., 2020).

In more detail, our contributions are as follows.

- We introduce a novel MANN architecture, namely the reservoir stack machine (RSM), which combines an echo state network (Jaeger and Haas, 2004) with an explicit stack to store inputs as well as auxiliary symbols.
- We prove that RSMs are at least as powerful as LR(1)-automata and thus can recognize all deterministic context-free languages (Knuth, 1965), whereas past reservoir memory machines (Paaßen et al., 2020) cannot.
- We evaluate our model on three benchmarks for Neural Turing Machines (latch, copy, and repeat copy) and on six context-free languages, verifying that it can learn all the tasks in a few seconds of training time from only 100 training sequences. By contrast, we also show that deep models (namely GRUs (Cho et al., 2014), the deep stack model of Suzgun et al. (Suzgun et al., 2019), and an end-to-end trained version of our proposed model) need much longer training times and sometimes fail to learn the task.

We note that our contributions are largely conceptual/theoretical. We do not expect that our model is widely applicable to practical tasks because required teaching signals for the dynamics might be missing. Instead, we investigate the interface between theoretical computer science and reservoir computing, providing further insight into the computational capabilities of neural networks, especially reservoir neural networks.

We begin our paper by describing background knowledge as well as related work (refer to Section 2), then introducing our model (refer to Section 3), and finally describing the experimental evaluation (refer to Section 4).

## 2 Background and Related Work

Our work in this paper is connected to several established lines of research. Due to the breadth of connected fields, we can only touch upon key ideas. Readers are invited to follow our references to more detailed discussions of the underlying topics. We begin with the central topic of our investigation, namely the computational power of neural networks, connect this to grammatical inference and memory-augmented neural nets, before introducing the underlying concepts of our reservoir stack machine model, namely LR(1) automata and echo state networks.

### 2.1 Computational Power of Recurrent Neural Networks

Computational power is typically measured by comparing a computational system to a reference automaton class in the Chomsky hierarchy, that is finite state automata (Chomsky-3), LR automata (Chomsky-2), linearly space-bounded Turing machines (Chomsky-1), or Turing machines (Chomsky-0) (Spersneider and Hammer, 1996). For the sake of space we do not define every automaton class in detail. We merely remark that, for our purpose, any automaton implements a function  $\mathcal{A} : \Sigma^* \rightarrow \{0, 1\}$  mapping input sequences  $\bar{x}$  over some set  $\Sigma$  to zero or one. If  $\mathcal{A}(\bar{x}) = 1$  we say that the automaton *accepts* the word. Otherwise, we say that it *rejects* the word. Accordingly, we define the *language*  $\mathcal{L}(\mathcal{A})$  accepted by the automaton as the set  $\mathcal{L}(\mathcal{A}) := \{\bar{x} \in \Sigma^* | \mathcal{A}(\bar{x}) = 1\}$ .

In this paper, we are interested in the computational power of neural networks. We particularly focus on recurrent neural networks (RNNs), which we define as follows.

**Definition 1** (Recurrent neural network). A *recurrent neural network* with  $n$  inputs,  $m$  neurons, and  $k$  outputs is defined as a 4-tuple  $(\mathbf{U}, \mathbf{W}, \sigma, g)$ , where  $\mathbf{U} \in \mathbb{R}^{m \times n}$ ,  $\mathbf{W} \in \mathbb{R}^{m \times m}$ ,  $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ , and  $g : \mathbb{R}^m \rightarrow \mathbb{R}^k$ .

Now, let  $\vec{x}_1, \dots, \vec{x}_T$  be a sequence with  $T$  elements over  $\mathbb{R}^n$ . We define the *state*  $h_t \in \mathbb{R}^m$  and the output  $y_t \in \mathbb{R}^k$  at time  $t$  via the equations:

$$\vec{h}_t = \sigma(\mathbf{U} \cdot \vec{x}_t + \mathbf{W} \cdot \vec{h}_{t-1}), \quad (1)$$

$$\vec{y}_t = g(\vec{h}_t), \quad (2)$$

where  $\sigma$  is applied element-wise and  $\vec{h}_0 = \vec{0}$ .

Now, let  $\Sigma \subset \mathbb{R}^n$  be some set and  $\mathcal{A} : \Sigma^* \rightarrow \{0, 1\}$  be the function implemented by an automaton. We say that an RNN *simulates* the automaton if for any input sequence  $\vec{x}_1, \dots, \vec{x}_T \in \Sigma^*$  we obtain  $\vec{y}_T = \mathcal{A}(\vec{x}_1, \dots, \vec{x}_T)$ . Early work has already demonstrated that recurrent neural networks (RNNs) with integer weights are sufficient to simulate finite state automata (Alon et al., 1991), that RNNs with rational weights can simulate Turing machines (Siegelmann and Sontag, 1995), and that RNNs with real weights even permit super-Turing computations (Siegelmann, 1995). Recently, Šíma has shown that binary RNNs with 1 or 2 extra rational-valued neurons lie in between Chomsky-3 and Chomsky-0, each recognizing some but not all languages in the in-between classes (Šíma, 2020).

Importantly, all these results rely on deterministic constructions to transform a known automaton into a neural network with carefully chosen weights. Learning languages from examples is a much more difficult task, studied under the umbrella of grammatical inference.

### 2.2 Grammatical inference

Grammatical inference (or grammar induction) is concerned with inferring a parser for a language just from examples for words that should be accepted (positive examples) and/or words

that should be rejected (negative examples) (de la Higuera, 2010). In general, this is a hard problem. For example, even some Chomsky-3 languages can not be learned from positive examples alone (de la Higuera, 2010). Further, grammar inference is typically ambiguous, i.e. there may be infinitely many parsers which recognize the same language - and selecting the 'smallest' among them is typically NP-hard (de la Higuera, 2010; Sperschneider and Hammer, 1996). Despite these difficulties, impressive progress has been made. For example, the Omphalos challenge has sparked new research into learning deterministic context-free grammars (DCFGs) (Clark, 2007). (Kim et al., 2019a; Shibata and Yoshinaka, 2016) have shown that special subclasses of probabilistic context-free grammars are learnable from examples. Finally, (Dyer et al., 2016; Kim et al., 2019b; Li et al., 2019; Shen et al., 2019; Yogatama et al., 2017) have suggested neural models to learn grammatical structure from data. Our work in this paper is less general than grammar inference approaches because we require training data on the desired parsing behavior for (short) examples instead of learning this behavior from scratch. However, our approach is more general than the deterministic translation schemes mentioned in the previous section because we do not require a full parser specification - only its behavior on a set of training data. As such, our approach lies between constructive proofs of computational power and grammatical inference. In a sense, our scenario resembles imitation learning, where a teacher demonstrates the correct (or at least viable) actions on a few training data instances and the models task is to generalize this behavior to the entire space of possible inputs (Ross and Bagnell, 2010).

Most importantly, though, our focus does not lie on inferring grammatical structure, but, rather, on studying the effect of a memory augmentation on computational power.

### 2.3 Memory-Augmented Neural Networks

For the purpose of this paper, we define a memory-augmented neural network (MANN) as any neural network that is extended with an explicit write and read operation to interact with an external memory. Recently, such models have received heightened interest, triggered by the Neural Turing Machine (Graves et al., 2016). The Neural Turing Machine implements a content-based addressing scheme, i.e. memory addresses are selected by comparing content in memory to a query vector and assigning attention based on similarity. The memory read is then given as the average of all memory lines, weighted by attention. Writing occurs with a similar mechanism. To support location-based addressing, the authors introduce the concept of a linkage matrix which keeps track of the order in which content was written to memory such that the network can decide to access whichever content was written after or before a queried line in memory. Further research has revised the model with sparser memory access (Rae et al., 2016), refined sharpening operations (Csordás and Schmidhuber, 2019), or more efficient initialization schemes (Collier and Beel, 2018). However, it remains that MANNs are difficult to train, often requiring tens of thousands of training epochs to converge even for simple memory access tasks (Collier and Beel, 2018). In this paper, we try to simplify this training by employing echo state networks (see below).

More precisely, we focus on a certain kind of MANN, namely stack-based models. Interestingly, such models go back far longer than the Neural Turing Machine, at least as far as the Neural State Pushdown Automaton of Giles et al. (1989). Such a model involves differentiable push and pop operations to write content onto the stack and remove it again. Recently, Suzgun et al. have built upon this concept and introduced stack recurrent neural networks (Suzgun et al., 2019) which can learn typical context free languages. A stack model is particularly appealing because it reduces the interaction with memory to two very basic operations, namely to push one single piece of information onto the stack, and to pop one single piece of information from it - no sharpening operations or linking matrices required. Further, despite

its simplicity, a stack is sufficient to implement LR(1)-automata, which we discuss next.

## 2.4 LR(1)-automata and the computational power of stacks

We define an LR(1)-automaton as follows.

**Definition 2.** An *LR(1)-automaton* is defined as a 4-tuple  $\mathcal{A} = (\Phi, \Sigma, R, F)$ , where  $\Phi$  and  $\Sigma$  are both finite sets with  $\Phi \cap \Sigma = \emptyset$ , where  $F \subseteq \Phi$ , and where  $R$  is a list of 4-tuples of the form  $(\bar{s}, x, j, A)$  with  $\bar{s} \in (\Phi \cup \Sigma)^*$ ,  $x \in \Sigma \cup \{\varepsilon\}$ ,  $j \in \mathbb{N}_0$ , and  $A \in \Phi$ .  $\varepsilon$  denotes the empty word. We call  $\Phi$  the nonterminal symbols,  $\Sigma$  the terminal symbols,  $R$  the rules, and  $F$  the accepting nonterminal symbols of the automaton.

In a slight abuse of notation, we also use the symbol  $\mathcal{A}$  to denote the function  $\mathcal{A} : \Sigma^* \rightarrow \{0, 1\}$  of the automaton, which we define as the result of Algorithm 1 for the automaton  $\mathcal{A}$  and the input  $\bar{w} \in \Sigma^*$ . We define the *accepted language*  $\mathcal{L}(\mathcal{A})$  of  $\mathcal{A}$  as the set  $\mathcal{L}(\mathcal{A}) = \{\bar{w} \in \Sigma^* \mid \mathcal{A}(\bar{w}) = 1\}$ . Finally, we define the set LR(1) as the set of all languages  $\mathcal{L}$  that can be accepted by some LR(1)-automaton.

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**Algorithm 1** The parsing algorithm for LR(1)-automata. The  $\exists$  quantifier in line 4 always chooses the first matching rule in the list  $R$ .

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1: function PARSE(automaton  $\mathcal{A} = (\Phi, \Sigma, R, F)$ , word  $w_1, \dots, w_T \in \Sigma^*$ )
2:   Initialize an empty stack  $\mathcal{S}$ .
3:   for  $y \leftarrow w_1, \dots, w_T, \#$  do
4:     while  $\exists (\bar{s}, x, j, A) \in R : \bar{s}$  is suffix of  $\mathcal{S}$  and  $x \in \{y, \varepsilon\}$  do
5:       Pop  $j$  elements from  $\mathcal{S}$ .
6:       Push  $A$  onto  $\mathcal{S}$ .
7:     end while
8:     Push  $y$  onto  $\mathcal{S}$ .
9:   end for
10:  if  $\exists A \in F : \mathcal{S} = A\#$  then
11:    return 1.
12:  else
13:    return 0.
14:  end if
15: end function

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Algorithm 1 works as follows: In each iteration, we consider one letter  $y$  of the input. Then, we check all rules  $(\bar{s}, x, j, A) \in R$  in order. In particular, we check if  $\bar{s}$  is a suffix of the current stack and if  $x$  is either  $y$  or  $\varepsilon$ . If both conditions hold, we say that the rule *matches*. In that case, we pop the top  $j$  elements from the stack and instead push the nonterminal  $A$  onto the stack. Once no rule matches anymore, we push  $y$  onto  $\mathcal{S}$  and continue. The last letter we process is a special end symbol  $\#$ . If only  $A\#$  is left on the stack for some nonterminal  $A \in F$ , we accept the word and return 1. Otherwise, we reject the word and return 0.

As an example, consider the language  $\mathcal{L}_{a^n b^n} := \{ab, aabb, aaabbb, \dots\}$ . This language is accepted by following LR(1)-automaton:

$$\mathcal{A}_{a^n b^n} = \left( \{\mathcal{S}\}, \{a, b\}, [(aSb, \varepsilon, 3, S), (a, b, 0, S)], \{\mathcal{S}\} \right), \quad (3)$$

where  $[]$  denotes an ordered list. Now, consider the word  $aabb$ . When we apply Algorithm 1, we notice that no rule matches for the first and second input letter. Accordingly, the first three states of the stack are  $\varepsilon$  (the empty stack),  $a$ , and  $aa$ . At this point, the current input

symbol is  $y = b$ . Now, the rule  $(a, b, 0, S)$  matches and, hence, the next stack state is  $aaS$ . Since no rule matches anymore, we proceed and obtain the next stack state  $aaSb$ . Now, the rule  $(aSb, \varepsilon, 3, S)$  matches and we obtain the stack state  $aS$ . Again, no rule matches anymore and we proceed to  $aSb$ . Now, rule  $(aSb, \varepsilon, 3, S)$  matches again and we obtain the stack  $S$  and the loop in lines 3-9 completes with the stack  $S\#$ . Finally, because  $S \in F$ , we obtain the output 1.

It is well known that the computational power of LR(1)-automata lies between Chomsky-3 and Chomsky-2 and corresponds exactly to deterministic context-free languages.

**Theorem 1.** *Chomsky-3  $\not\subseteq$  LR(1) = DCFG  $\not\subseteq$  Chomsky-2.*

For a proof, refer to standard text books such as Sperschneider and Hammer (1996).

Interestingly, if we provide a model with a second stack, this is sufficient to simulate an entire Turing machine, raising the computational power to Chomsky-0 (Sperschneider and Hammer, 1996; Suzgun et al., 2019). Overall, stacks appear as a particularly promising datastructure to augment neural nets with more computational power. In this paper, we augment a neural net with a parsing scheme similar to Algorithm 1. The underlying neural network is an echo state network.

## 2.5 Echo State Networks

We define an echo state network as follows.

**Definition 3** (Reservoir and echo state network). We define a *reservoir* with  $n$  inputs and  $m$  neurons as a triple  $\mathcal{R} = (\mathbf{U}, \mathbf{W}, \sigma)$  where  $\mathbf{U} \in \mathbb{R}^{m \times n}$ ,  $\mathbf{W} \in \mathbb{R}^{m \times m}$ , and  $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ .

We say that a reservoir is *compact* with respect to a compact set  $\Sigma \subset \mathbb{R}^n$  if there exists a compact set  $\mathcal{H} \subset \mathbb{R}^m$  such that  $\vec{0} \in \mathcal{H}$  and for all  $\vec{h} \in \mathcal{H}$  as well as all  $\vec{x} \in \Sigma$  it holds:  $\sigma(\mathbf{W} \cdot \vec{h} + \mathbf{U} \cdot \vec{x}) \in \mathcal{H}$ .

We further say that a reservoir fulfills the *echo state property* with respect to a compact set  $\Sigma \subset \mathbb{R}^n$  if it is compact with respect to  $\Sigma$  and there exists an infinite null-sequence  $\delta_1, \delta_2, \dots$ , such that for any infinite input sequence  $\vec{x}_1, \vec{x}_2, \dots$  over  $\Sigma$  and any two initial states  $\vec{h}_0, \vec{h}'_0 \in \mathcal{H}$ , it holds:  $\|\vec{h}_t - \vec{h}'_t\| \leq \delta_t$ , where  $\vec{h}_t = \sigma(\mathbf{W} \cdot \vec{h}_{t-1} + \mathbf{U} \cdot \vec{x}_t)$  and  $\vec{h}'_t = \sigma(\mathbf{W} \cdot \vec{h}'_{t-1} + \mathbf{U} \cdot \vec{x}_t)$  for all  $t \in \mathbb{N}$ .

Now, let  $\Sigma \subset \mathbb{R}^n$  be a compact set. We define an *echo state network* (ESN) over  $\Sigma$  with  $m$  neurons as a recurrent neural network  $(\mathbf{U}, \mathbf{W}, \sigma, g)$  where  $(\mathbf{U}, \mathbf{W}, \sigma)$  is a reservoir with  $n$  inputs and  $m$  neurons that fulfills the echo state property with respect to  $\Sigma$ .

Note that our definition of the echo state property slightly deviates from the original definition, but is provably equivalent (Yildiz et al., 2012).

Intuitively, the echo state property guarantees that the state  $\vec{h}_t$  of a reservoir represents the past via a fractal-like encoding that is dominated by its suffix (Tino and Dorffner, 2001). This ensures that the reservoir reacts similarly to all sequences that share the same suffix and thus can be viewed as a Markovian filter (Gallicchio and Micheli, 2011). Importantly, the specific parameters of the network do not matter much. As long as the echo state property holds, the reservoir state provides a representation of the recent past and, thus, we only need to adapt the output function  $g$  to the task at hand, without any change to the reservoir. This makes echo state networks appealing, because  $g$  can usually be trained quickly using convex optimization, whereas the adaptation of  $\mathbf{U}$  and  $\mathbf{W}$  requires heuristic schemes (Jaeger and Haas, 2004). This is also our main motivation for using reservoir computing: They are a promising starting point for networks that provide computational capabilities but remain fast to train.

However, the echo state property also limits the computational power of ESNs to languages that can be recognized by a finite suffix (i.e. definite memory machines; Hammer and Tiño,

2003). For example, consider the language  $ab^*$ . For any  $T \in \mathbb{N}$ , this language contains the string  $ab^T$  but not the string  $b^T$ . However, the echo state property enforces that the reservoir states for these two strings become eventually indistinguishable, meaning that no (uniformly continuous) output function  $g$  can map them to different values. It is currently an open question whether echo state networks that operate at the edge of chaos extend this capability (Boedecker et al., 2012; Farkaš et al., 2016; Gallicchio and Micheli, 2011; Gallicchio et al., 2018). The provable computational limitations of ESNs make them a particularly interesting object of study for memory-augmented neural networks because any computational capability beyond the limits of definite memory machines (Hammer and Tiño, 2003) must provably be due to the augmentation.

We note that our prior work has already investigated the computational effect of some memory augmentations to ESNs. In particular, this paper is an extension of the ESANN 2020 Paper “Reservoir Memory Machines” (Paaßen and Schulz, 2020). In contrast to this prior work, we do not use a constant-sized memory but instead a stack, and we prove that this stack improves computational power to that of LR(1) automata. In a separate and distinct extension (Paaßen et al., 2020), we kept the constant-sized memory but simplified the memory access behavior and added an associative memory access mechanism. Such networks can provably simulate any finite state automaton. However, we prove later in Section 3.1 that this memory does not suffice to simulate general LR(1) automata. Instead, we use a stack as memory architecture and prove that such a network can simulate all LR(1) automata. Finally, the experimental evaluation for both papers are mostly disjoint, with Paaßen et al. (2020) focusing on associative memory and recall tasks as well as finite state machines, whereas, in this paper, we focus on LR(1) language recognition.

### 3 Method

In this section, we introduce our proposed model, the reservoir stack machine. First, we define the dynamics of our model and explain its intended mechanism. Then, in Section 3.1, we prove that a reservoir stack machine with a sufficiently rich reservoir can simulate any LR(1)-automaton, while the reservoir memory machine (Paaßen et al., 2020) cannot. Finally, we provide a training scheme in Section 3.2.

We construct a reservoir stack machine (RSM) as a combination of an echo state network (ESN) (Jaeger and Haas, 2004) with a stack, where the stack is controlled similarly to an LR(1)-automaton (Knuth, 1965). In more detail:

**Definition 4.** Let  $\Sigma \subset \mathbb{R}^n$  and  $\Phi \subset \mathbb{R}^n$  be compact sets with  $\Sigma \cap \Phi = \emptyset$  and  $\vec{0} \notin \Phi$ . We define a reservoir stack machine (RSM) over  $\Sigma$  and  $\Phi$  with  $m$  neurons,  $J$  maximally popped symbols, and  $L$  outputs as a 7-tuple  $\mathcal{M} = (\mathbf{U}, \mathbf{W}, \sigma, c^{\text{pop}}, c^{\text{push}}, c^{\text{shift}}, c^{\text{out}})$  where  $(\mathbf{U}, \mathbf{W}, \sigma)$  is a reservoir with  $n$  inputs and  $m$  neurons that conforms to the echo state property on  $\Sigma \cup \Phi$ , and where  $c^{\text{pop}} : \mathbb{R}^{2m} \rightarrow \{0, \dots, J\}$ ,  $c^{\text{push}} : \mathbb{R}^{2m} \rightarrow \Phi \cup \{\vec{0}\}$ ,  $c^{\text{shift}} : \mathbb{R}^{2m} \rightarrow \{0, 1\}$ , and  $c^{\text{out}} : \mathbb{R}^{2m} \rightarrow \mathbb{R}^L$ .

Let  $\vec{x}_1, \dots, \vec{x}_T$  be a sequence over  $\Sigma$ . We define the output sequence  $\mathcal{M}(\vec{x}_1, \dots, \vec{x}_T) = \vec{y}_1, \dots, \vec{y}_{T+1}$  as the output of Algorithm 2 for the input  $\vec{x}_1, \dots, \vec{x}_T$ . We say that an RSM simulates an automaton  $\mathcal{A}$  if  $y_{T+1} = \mathcal{A}(\vec{x}_1, \dots, \vec{x}_T)$  for any input sequence  $\vec{x}_1, \dots, \vec{x}_T \in \Sigma^*$ .

Roughly speaking, Algorithm 2 works as follows. In every iteration  $t$  of the RSM, we represent the input sequence  $\vec{x}_1, \dots, \vec{x}_t$  up to time  $t$  with a state vector  $\vec{h}_t$ , and the stack content  $\vec{s}_1, \dots, \vec{s}_\tau$  with a state vector  $\vec{g}_t$ , both using the same reservoir  $(\mathbf{U}, \mathbf{W}, \sigma)$ . Based on the concatenated state vector  $(\vec{h}_t, \vec{g}_t)$ , we then let  $c^{\text{pop}}$  decide how many symbols to pop from the stack and we let  $c^{\text{push}}$  decide which symbol (if any) to push on the stack until both

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**Algorithm 2** The dynamics of a reservoir stack machine  $\mathcal{M} = (\mathbf{U}, \mathbf{W}, \sigma, c^{\text{pop}}, c^{\text{push}}, c^{\text{shift}}, c^{\text{out}})$  on the input sequence  $\vec{x}_1, \dots, \vec{x}_T$ .

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1: function RSM(Input sequence  $\vec{x}_1, \dots, \vec{x}_T$ )
2:   Set  $\vec{x}_{T+1} \leftarrow \vec{0}$ .
3:   Initialize an empty stack  $\mathcal{S}$ .
4:   Initialize  $\vec{h}_0 \leftarrow \vec{0}, \vec{g}_1 \leftarrow \vec{0}$ .
5:   for  $t \leftarrow 1, \dots, T + 1$  do
6:      $\vec{h}_t \leftarrow \sigma(\mathbf{U} \cdot \vec{x}_t + \mathbf{W} \cdot \vec{h}_{t-1})$ .
7:     while True do
8:        $j \leftarrow c^{\text{pop}}(\vec{h}_t, \vec{g}_t)$ .
9:       if  $j > 0$  then
10:        Pop  $j$  elements from  $\mathcal{S}$ .
11:         $\vec{g}_t \leftarrow \vec{0}$ .
12:        for  $\vec{s} \leftarrow$  elements of  $\mathcal{S}$  do
13:           $\vec{g}_t \leftarrow \sigma(\mathbf{U} \cdot \vec{s} + \mathbf{W} \cdot \vec{g}_t)$ .
14:        end for
15:      end if
16:       $\vec{a} \leftarrow c^{\text{push}}(\vec{h}_t, \vec{g}_t)$ .
17:      if  $\vec{a} \neq \vec{0}$  then
18:        Push  $\vec{a}$  onto  $\mathcal{S}$ .
19:         $\vec{g}_t \leftarrow \sigma(\mathbf{U} \cdot \vec{a} + \mathbf{W} \cdot \vec{g}_t)$ .
20:      end if
21:      if  $j = 0$  and  $\vec{a} = \vec{0}$  then
22:        Break.
23:      end if
24:    end while
25:     $\vec{y}_t \leftarrow c^{\text{out}}(\vec{h}_t, \vec{g}_t)$ .
26:    if  $c^{\text{shift}}(\vec{h}_t, \vec{g}_t) > 0$  then
27:      Push  $\vec{x}_t$  onto  $\mathcal{S}$ .
28:       $\vec{g}_{t+1} \leftarrow \sigma(\mathbf{U} \cdot \vec{x}_t + \mathbf{W} \cdot \vec{g}_t)$ .
29:    else
30:       $\vec{g}_{t+1} \leftarrow \vec{g}_t$ .
31:    end if
32:  end for
33:  return  $\vec{y}_1, \dots, \vec{y}_{T+1}$ .
34: end function

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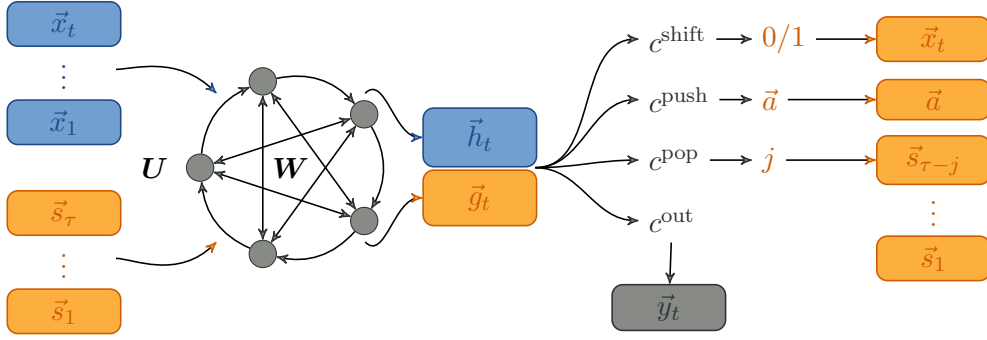


Figure 1: An illustration of the reservoir stack machine dynamics. The input sequence up to time  $t$  is represented as a state  $\vec{h}_t$  via the reservoir  $(\mathbf{U}, \mathbf{W}, \sigma)$  and the current stack is represented as a state  $\vec{g}_t$  via the same reservoir. The concatenated state  $(\vec{h}_t, \vec{g}_t)$  is plugged into classifiers  $c^{\text{shift}}$ ,  $c^{\text{push}}$ , and  $c^{\text{pop}}$ , which generate the next stack state (right). Note that  $c^{\text{pop}}$  and  $c^{\text{push}}$  are called repeatedly until they both return 0. The output function  $c^{\text{out}}$  generates the next output  $\vec{y}_t$ .

classifiers return 0. Then, we let a function  $c^{\text{out}}$  decide the current output  $\vec{y}_t$ , and we let a binary classifier  $c^{\text{shift}}$  decide whether to push the current input symbol  $\vec{x}_t$  onto the stack or not. Every time we change the stack, we update the state  $\vec{g}_t$  accordingly. Refer to Figure 1 for a graphical illustration.

As an example, let us construct an RSM that simulates the LR(1)-automaton  $\mathcal{A}_{a^n b^n}$  in Equation 3. For such a simulation, we only need to distinguish three cases: First, if the top of the stack is  $b$ , we need to pop 3 symbols from the stack and push the symbol  $S$ ; second, if the top of the stack is  $a$  and the current input symbol is  $b$ , we need to pop 0 symbols from the stack and push the symbol  $S$ ; third, in all other cases we neither pop nor push. Distinguishing these cases is easy with an echo state network. For example, consider a cycle reservoir with jumps (CRJ) (Rodan and Tiño, 2012) with five neurons, jump length 2, input weight 1 and cycle weight 0.5, i.e. the architecture displayed in Figure 1. Figure 2 shows a two-dimensional principal component analysis of the states generated by this CRJ for one-hot-coding sequences of length up to 10, where color and shape indicate the last symbol of the sequence. As we can see, the states are linearly separable based on their last symbol. Accordingly, we can implement  $c^{\text{pop}}$  and  $c^{\text{push}}$  as linear classifiers,  $c^{\text{shift}}$  as a constant 1, and  $c^{\text{out}}$  merely needs to recognize whether the current stack content is exactly  $S$  or anything else. Then, our reservoir stack machine will return 1 for any word that is in  $\mathcal{L}_{a^n b^n}$  and 0 otherwise.

This example illustrates how the reservoir stack machine is intended to work: The input state  $\vec{h}_t$  represents the current input symbol and the stack state  $\vec{g}_t$  a suffix of the stack, such that simple classifiers can make the decision which rule of an LR(1)-automaton to apply. Next, we generalize this idea to a theorem, showing how to simulate an LR(1)-automaton with an RSM more generally.

### 3.1 Theory

This section has two goals: First, we wish to show that an echo state network with a constant-sized memory as proposed in Paaßen et al. (2020) is not sufficient to simulate general LR(1) automata. Second, we wish to show that the reservoir stack machine is sufficient, provided that the underlying reservoir is sufficiently rich.

We begin with the first claim. First, we recall the definition of a reservoir memory machine

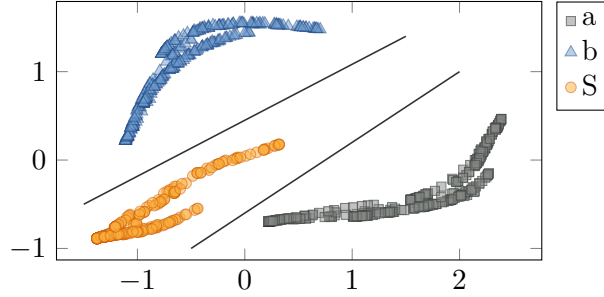


Figure 2: A 2-dimensional PCA of the representations produced by the reservoir in Figure 1 for stacks up to length 10. The color and shape represents the symbol on top of the stack. Lines indicate the linear separability of representations according to the top symbol.

from Paaßen et al. (2020) (in a slightly adapted form for our notation in this paper).

**Definition 5** (Reservoir Memory Machine (Paaßen et al., 2020)). Let  $\Sigma \subset \mathbb{R}^n$  be a compact set. We define a reservoir memory machine (RMM) over  $\Sigma$  with  $m$  neurons,  $K$  rows of memory and  $L$  outputs as a 5-tuple of the form  $\mathcal{M} = (\mathbf{U}, \mathbf{W}, \sigma, c, g)$ , where  $(\mathbf{U}, \mathbf{W}, \sigma)$  is a reservoir with  $n$  inputs and  $m$  neurons that fulfills the echo state property with respect to  $\Sigma$ , where  $c: \mathbb{R}^m \rightarrow \{0, \dots, K\}$ , and where  $g: \mathbb{R}^m \rightarrow \mathbb{R}^L$ .

Let  $\vec{x}_1, \dots, \vec{x}_T \in \Sigma^*$ . We define the output sequence  $\mathcal{M}(\vec{x}_1, \dots, \vec{x}_T) = \vec{y}_1, \dots, \vec{y}_T$  as the output of Algorithm 3 for the input  $\vec{x}_1, \dots, \vec{x}_T$ .

---

**Algorithm 3** The dynamics of a reservoir memory machine  $\mathcal{M} = (\mathbf{U}, \mathbf{W}, \sigma, c, g)$  on the input sequence  $\vec{x}_1, \dots, \vec{x}_T$ .

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1: function RMM(Input sequence  $\vec{x}_1, \dots, \vec{x}_T$ )
2:   Initialize an empty memory matrix  $\mathbf{M}$  of size  $K \times m$  with zeros.
3:   Initialize  $\vec{h}_0$ .
4:   for  $t \leftarrow 1, \dots, T$  do
5:      $\vec{h}_t \leftarrow \sigma(\mathbf{W} \cdot \vec{h}_{t-1} + \mathbf{U} \cdot \vec{x}_t)$ .
6:      $a_t \leftarrow c(\vec{h}_t)$ .
7:     if  $a_t > 0$  then
8:       if  $\vec{m}_{a_t}$  has already been written to then
9:          $\vec{h}_t \leftarrow \vec{m}_{a_t}$ .
10:      else
11:         $\vec{m}_{a_t} \leftarrow \vec{h}_t$ .
12:      end if
13:    end if
14:     $\vec{y}_t \leftarrow g(\vec{h}_t)$ .
15:  end for
16:  return  $\vec{y}_1, \dots, \vec{y}_T$ .
17: end function

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Intuitively, the classifier  $c$  controls which memory location we are currently accessing and the function  $g$  controls the output of the system. As long as  $c$  outputs zero, the RMM behaves like a standard ESN. When it outputs a nonzero memory address  $a_t$  the first time at time  $t$ , we write the current state  $\vec{h}_t$  to the  $a_t$ th row of the memory. When the same address  $a_{t'}$  is accessed again at time  $t' > t$ , we override  $\vec{h}_{t'}$  with the memory entry at  $a_{t'}$ . In other words,

the output will be the same as that of an ESN until we have two times  $t, t'$  with  $t' > t$  such that  $a_t = a_{t'} > 0$ . As soon as that happens, the RMM recalls past states.

This memory mechanism is provably sufficient to simulate any finite state automaton (Paaßen et al., 2020). However, it is insufficient to simulate at least some LR(1) automata. In particular, consider the following LR(1) automaton:

$$\mathcal{A}_{\text{palin}} = \left( \{S\}, \{a, b, \$\}, [(\$ , \varepsilon, 1, S), (aSa, \varepsilon, 3, S), (bSb, \varepsilon, 3, S)], \{S\} \right). \quad (4)$$

This automaton recognizes the language  $\mathcal{L}(\mathcal{A}_{\text{palin}})$  of palindromes over  $a$  and  $b$  with  $\$$  in the middle. We will now show that no RMM can simulate this automaton.

**Theorem 2.** *Let  $\Sigma = \{a, b, \$\}$  be a set of  $n$ -dimensional vector encodings of the symbols  $a, b$  and  $\$$ . Further, let  $\mathcal{M} = (\mathbf{U}, \mathbf{W}, \sigma, c, g)$  be a reservoir memory machine with  $m$  neurons,  $K$  rows of memory, and  $L = 1$  outputs where  $g$  is a uniformly continuous output function, i.e. for any  $\epsilon > 0$  there exists some  $\tilde{\delta}_\epsilon$  such that for any two  $\vec{h}, \vec{h}' \in \mathbb{R}^m$  with  $\|\vec{h} - \vec{h}'\| < \tilde{\delta}_\epsilon$  it holds  $|g(\vec{h}) - g(\vec{h}')| < \epsilon$ . Then,  $\mathcal{M}$  does not simulate  $\mathcal{A}_{\text{palin}}$ .*

*Proof.* We perform a proof by contradiction, i.e. we assume that there exists an RMM  $\mathcal{M}$  with a uniformly continuous output function  $g$  and which does simulate  $\mathcal{A}_{\text{palin}}$  and we show that this yields a contradiction.

Our proof has two steps. First, we show that any RMM that simulates  $\mathcal{A}_{\text{palin}}$  never visits line 9 of Algorithm 3 for inputs of the form  $a^T \$ a^T$  or  $ba^{T-1} \$ a^{T-1} b$ , i.e. it behaves like an ESN for these inputs. Then, we show that a reservoir with the echo state property can not distinguish the inputs  $a^T \$ a^T$  and  $ba^T \$ a^T$  for large enough  $T$ , which in turn means that an ESN with uniformly continuous output function can not distinguish them, either.

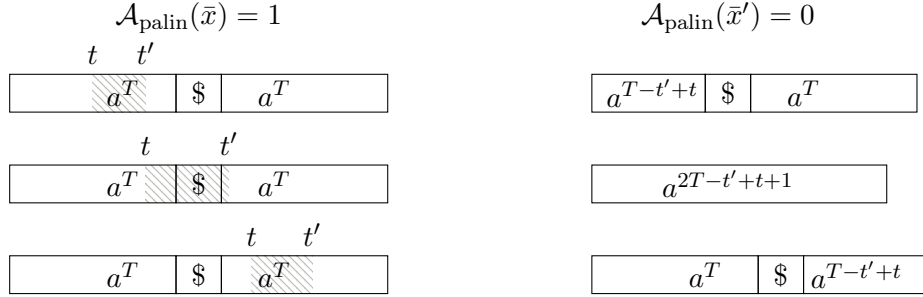
Regarding our first claim, assume that line 9 of of Algorithm 3 is visited. Then, there must exist  $t, t'$  such that  $0 < t < t' \leq 2T + 1$  and  $a_t = a_{t'} > 0$ . Now, let  $t'$  be as large as possible and  $t$  be as small as possible. In that case, the original sequence  $\vec{x}$  and  $\vec{x}'$  with the subsequence from  $t$  to  $t'$  removed yield the same state (refer to Paaßen et al. (2020)) and, accordingly, the same output. However, we can show that  $\vec{x}$  would be accepted by  $\mathcal{A}_{\text{palin}}$  while  $\vec{x}'$  would not be accepted. Accordingly, the RMM does not simulate  $\mathcal{A}_{\text{palin}}$ , which is a contradiction. Refer to Figure 3 for an illustration of all possible choices of  $t$  and  $t'$ .

By extension of this argument, we also know that line 9 is not visited for the prefix  $ba^T \$ a^T$ , otherwise it would also be visited for  $ba^T \$ a^T b$ . Further, given that line 9 is not visited, the output generated by Algorithm 3 is the same as the output of the ESN  $(\mathbf{U}, \mathbf{W}, \sigma, g)$  according to Definition 3, because lines 6-8 and 10-13 neither influence the state  $\vec{h}_t$  nor the output.

Next, we show that the ESN  $(\mathbf{U}, \mathbf{W}, \sigma, g)$  can not simulate the automaton  $\mathcal{A}_{\text{palin}}$ . For this purpose, we consider the output function  $g$  and the echo state property in a bit more detail. Because  $g$  is uniformly continuous, we know that there must exist some  $\tilde{\delta}$ , such that for any two  $\vec{h}, \vec{h}' \in \mathbb{R}^m$  with  $\|\vec{h} - \vec{h}'\| < \tilde{\delta}$  we obtain  $|g(\vec{h}) - g(\vec{h}')| < 1$ . Next, let  $\delta_1, \delta_2, \dots$  be the null sequence from the definition of the echo state property (refer to Definition 3). Because this is a null sequence, there must exist some  $t^*$  such that for any  $t > t^*$  we obtain  $\delta_t < \tilde{\delta}$ . Now, consider the two inputs  $a^T \$ a^T$  and  $ba^T \$ a^T$  with  $T = \lceil t^*/2 \rceil$  and consider the two initial states  $\vec{h}_0$  and  $\vec{h}'_0 = \sigma(\mathbf{U} \cdot b)$ , as well as the continuations  $\vec{h}_t = \sigma(\mathbf{W} \cdot \vec{h}_{t-1} + \mathbf{U} \cdot \vec{x}_t)$  and  $\vec{h}'_t = \sigma(\mathbf{W} \cdot \vec{h}'_{t-1} + \mathbf{U} \cdot \vec{x}_t)$  for all  $t \in \{1, \dots, 2T + 1\}$  where  $\vec{x}_1, \dots, \vec{x}_{2T+1} = a^T \$ a^T$ .

The echo state property now guarantees that  $\|\vec{h}_{2T+1} - \vec{h}'_{2T+1}\| \leq \delta_{2T+1} < \tilde{\delta}$  because  $2T + 1 > t^*$ . Accordingly, the uniform continuity of  $g$  guarantees that  $|g(\vec{h}_{2T+1}) - g(\vec{h}'_{2T+1})| < 1$ . However, we have  $|\mathcal{A}_{\text{palin}}(ba^T \$ a^T) - \mathcal{A}_{\text{palin}}(a^T \$ a^T)| = 1$ . Accordingly,  $\mathcal{M}$  does not simulate  $\mathcal{A}_{\text{palin}}$ .  $\square$

Construction for  $a^T\$a^T$



Construction for  $ba^{T-1}\$a^{T-1}b$

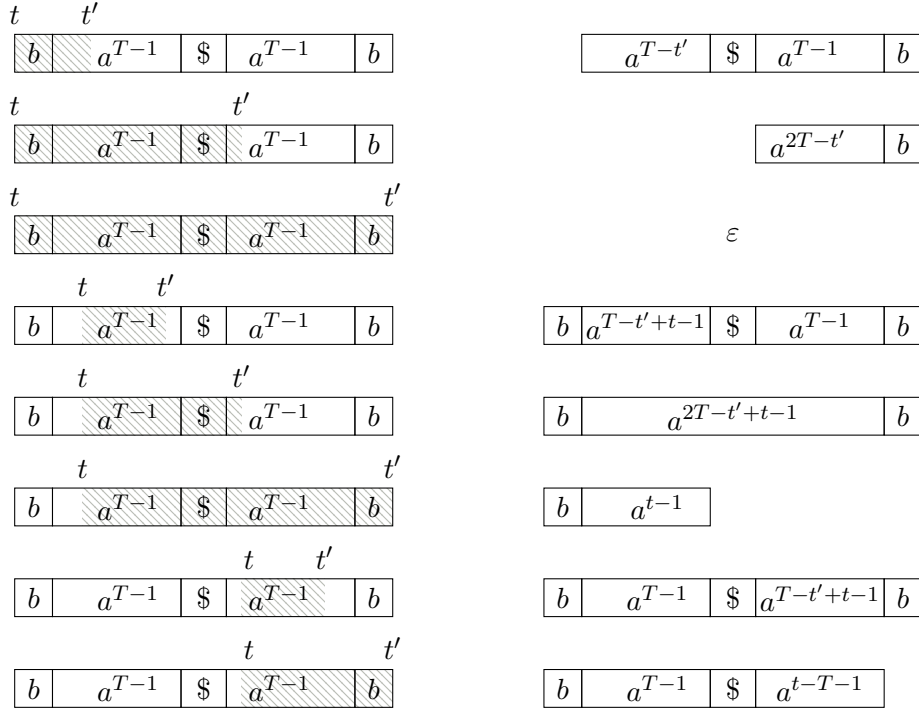


Figure 3: All possibilities to recall memory in an RMM at different locations of processing inputs of the form  $a^T\$a^T$  (top) or  $ba^{T-1}\$a^{T-1}b$  (bottom). In all inputs on the left, the shaded region indicates the portion of the input that is ignored and the input on the right corresponds to a version of the input where the shaded region is removed. In all cases, we observe that the input on the left would be accepted by  $\mathcal{A}_{\text{palin}}$ , while the input on the right would not. This is a contradiction, because both inputs would receive the same state (and hence the same output) by the RMM.

Our next goal is to prove that an RSM with a sufficiently rich reservoir can simulate any LR(1)-automaton, only by adjusting the functions  $c^{\text{pop}}$ ,  $c^{\text{push}}$ , and  $c^{\text{out}}$  and  $c^{\text{shift}}$ . To make this precise, we first define formally what we mean by a 'sufficiently rich reservoir' and then go on to our main theorem.

**Definition 6** ( $\bar{w}$ -separating reservoirs). Let  $\Sigma \subset \mathbb{R}^n$  be some finite set and  $\mathcal{R} = (\mathbf{U}, \mathbf{W}, \sigma)$  be a reservoir with  $n$  inputs and  $m$  neurons. We define the representation  $h_{\mathcal{R}}(\vec{x}_1, \dots, \vec{x}_T)$  of  $\vec{x}_1, \dots, \vec{x}_T \in \Sigma^*$  according to the reservoir  $\mathcal{R}$  as the state  $\vec{h}_T$  resulting from Equation 1.

Now, let  $\bar{w} \in \Sigma^*$ . We say that  $\mathcal{R}$  is a  $\bar{w}$ -separating reservoir if there exists an affine function  $f_{\bar{w}}(\vec{h}) = \mathbf{V} \cdot \vec{h} + b$  such that for all  $\bar{u} \in \Sigma^*$  it holds:  $f_{\bar{w}}(h_{\mathcal{R}}(\bar{u})) > 0$  if  $\bar{u}$  has  $\bar{w}$  as suffix and  $f_{\bar{w}}(h_{\mathcal{R}}(\bar{u})) \leq 0$  otherwise.

For example, Figure 2 shows the representations of random words  $\bar{w} \in \{a, b, S\}^*$  up to length 10. The figure shows that the reservoir is  $a$ -separating,  $b$ -separating, and  $S$ -separating. In general, reservoirs with contractive properties have a strong bias to be separating, because the suffix dominates the representation (Tino and Dorffner, 2001). We also use this separation property to simulate LR(1)-automata with RSMs.

**Theorem 3.** Let  $\mathcal{A} = (\Phi, \Sigma, R, F)$  be an LR(1)-automaton with  $\Phi \subset \mathbb{R}^n$  and  $\Sigma \subset \mathbb{R}^n$ . Further, let  $\mathcal{X} = \{x | (\bar{s}, x, j, A) \in R\}$ , and let  $\mathcal{S} = \{\bar{s} | (\bar{s}, x, j, A) \in R\}$ . Finally, let  $\mathcal{R} = (\mathbf{U}, \mathbf{W}, \sigma)$  be a reservoir over  $\Sigma \cup \Phi$  with  $n$  inputs and  $m$  neurons such that  $\mathcal{R}$  is a  $\bar{w}$ -separator for all  $\bar{w} \in \mathcal{X} \cup \mathcal{S}$  and such that  $\{h_{\mathcal{R}}(\bar{w}) | \bar{w} \in (\Phi \cup \Sigma)^*, \bar{w} \notin F\} \cap \{h_{\mathcal{R}}(A) | A \in F\} = \emptyset$ , i.e. the representations of accepting nonterminals and all other words are disjoint.

Then, we can construct classifiers  $c^{\text{pop}}$ ,  $c^{\text{push}}$ ,  $c^{\text{shift}}$ , and  $c^{\text{out}}$ , such that the reservoir stack machine  $(\mathbf{U}, \mathbf{W}, \sigma, c^{\text{pop}}, c^{\text{push}}, c^{\text{shift}}, c^{\text{out}})$  over  $\Sigma$  and  $\Phi$  simulates  $\mathcal{A}$ .

*Proof.* Per definition,  $\mathcal{R}$  is a  $\bar{w}$ -separator for every word  $\bar{w} \in \mathcal{X} \cup \mathcal{S}$ . Accordingly, there must exist affine functions  $f_{\bar{w}} : \mathbb{R}^m \rightarrow \mathbb{R}$ , such that for all  $\bar{u} \in \Sigma^*$  it holds:  $f_{\bar{w}}(h_{\mathcal{R}}(\bar{u})) > 1$  if  $\bar{w}$  is a suffix of  $\bar{u}$  and  $f_{\bar{w}}(h_{\mathcal{R}}(\bar{u})) \leq 0$  otherwise.

Now, we define the functions  $c^{\text{pop}}$  and  $c^{\text{push}}$  via the following procedure. We iterate over the rules  $(\bar{s}, x, j, A) \in R$  of the LR(1)-automaton in ascending order. If  $f_{\bar{s}}(\vec{g}_t) > 0$  and  $f_x(\vec{h}_t) > 0$ , we return  $c^{\text{pop}}(\vec{h}_t, \vec{g}_t) = j$  and  $c^{\text{push}}(\vec{h}_t, \vec{g}_t) = A$ . If this does not occur for any rule, we return  $c^{\text{pop}}(\vec{h}_t, \vec{g}_t) = 0$  and  $c^{\text{push}}(\vec{h}_t, \vec{g}_t) = \vec{0}$ .

We further define  $c^{\text{shift}}(\vec{h}_t, \vec{g}_t)$  as constant 1. Finally, we define  $c^{\text{out}}(\vec{h}_t, \vec{g}_t)$  as 1 if  $\vec{g}_t \in \{h_{\mathcal{R}}(A) | A \in F\}$  and as 0 otherwise. Note that all four functions are well-defined due to our separation requirements on the reservoir.

Our claim is now that the stack  $\mathcal{S}$  when arriving at line 25 in Algorithm 2 is the same as the stack  $\mathcal{S}$  when arriving at line 8 in Algorithm 1. We prove this by induction over the number of inner loop iterations. If no iterations have occurred, the stack is empty in both cases. Now, assume that the stack has a certain state  $\mathcal{S}$  in both Algorithm 1 and Algorithm 2 and we now enter line 4 in Algorithm 1 and line 8 in Algorithm 2, respectively.

First, consider the case that the  $\exists$  quantifier in line 4 of Algorithm 1 finds no matching rule, i.e. there is no rule  $(\bar{s}, x, j, A) \in R$  such that the current stack  $\mathcal{S}$  has the suffix  $\bar{s}$  and the current input symbol  $y$  equals  $x$  (or  $x = \varepsilon$ ). In that case,  $f_{\bar{s}}(\vec{g}_t) \leq 0$  or  $f_x(\vec{h}_t) \leq 0$ , which implies  $c^{\text{pop}}(\vec{h}_t, \vec{g}_t) = 0$  and  $c^{\text{push}}(\vec{h}_t, \vec{g}_t) = \vec{0}$ . Accordingly, lines 9-20 leave the stack unchanged and the condition in line 21 applies, such that we break out of the loop. Equivalently, the loop in Algorithm 1 stops.

Second, consider the case that at least one rule matches in line 4 of Algorithm 1. In that case, the first matching rule  $(\bar{s}, x, j, A)$  is selected (by definition in the Algorithm description). Accordingly, the current stack  $\mathcal{S}$  has the suffix  $\bar{s}$  and the current input symbol  $y$  equals  $x$  or  $x = \varepsilon$ . By virtue of the separation conditions on our reservoir,  $f_{\bar{s}}(\vec{g}_t) > 0$  and  $f_x(\vec{h}_t) >$

0. Therefore, by our definition of  $c^{\text{pop}}$  and  $c^{\text{push}}$  above, we obtain  $c^{\text{pop}}(\vec{h}_t, \vec{g}_t) = j$  and  $c^{\text{push}}(\vec{h}_t, \vec{g}_t) = A$ .

Next, note that lines 10 and 18 of Algorithm 2 update the stack just as lines 5-6 in Algorithm 1 do and that lines 11-14 as well as line 19 update the stack state  $\vec{g}_t$  accordingly.

Finally, note that the condition in line 26 of Algorithm 2 is always fulfilled because we defined  $c^{\text{shift}}(\vec{h}_t, \vec{g}_t) = 1$ . Hence, line 27 of Algorithm 2 updates the stack just as line 8 in Algorithm 1 does and line 25 updates the stack representation accordingly.

In conclusion, at every iteration of the computation, the stacks of Algorithms 1 and 2 are the same. It only remains to show that the last output  $\vec{y}_{T+1}$  is the same as  $\mathcal{A}(\bar{w})$ . If  $\mathcal{A}(\bar{w}) = 1$ , this means that the stack at the end of the computation in Algorithm 1 was  $\mathcal{S} = A\#$  for some accepting nonterminal  $A \in F$ . For Algorithm 2, this implies that the stack before the last push operation in line 24 must have been  $\mathcal{S} = A$ . Hence,  $c^{\text{out}}(\vec{h}_{T+1}, \vec{g}_{T+1}) = 1$ . Conversely, if  $\mathcal{A}(\bar{w}) = 0$ , the stack must have been something else, and hence  $c^{\text{out}}(\vec{h}_{T+1}, \vec{g}_{T+1}) = 0$ .  $\square$

We note that we assume linear separability of single suffices, but we may still require nonlinear classifiers for  $c^{\text{pop}}$ ,  $c^{\text{push}}$ , and  $c^{\text{out}}$  because these are defined over unions of suffices, which in turn may yield non-linear boundaries. As such, we recommend non-linear classifiers in practice, such as radial basis function SVMs.

This concludes our theory chapter. We now know that RMMs cannot recognize LR(1) languages in general, but RSMs can. Our next step is to describe how we can train an RSM.

### 3.2 Training

In this section, we describe how to train an RSM from example data. As training data, we require 5-tuples of the form  $(\bar{x}, \mathbf{J}, \mathbf{A}, \vec{\rho}, \mathbf{Y})$  where  $\bar{x} \in \Sigma^T$  is a sequence of  $T$  input vectors for some  $T$ ,  $\mathbf{J} \in \{0, \dots, J\}^{T+1 \times M}$  is a matrix of desired pop actions for some  $M$ ,  $\mathbf{A} \in \Phi^{T+1 \times M}$  is a tensor of desired push actions,  $\vec{\rho} \in \{0, 1\}^{T+1}$  is a vector of desired shift actions, and  $\mathbf{Y} \in \mathbb{R}^{T+1 \times L}$  is a matrix of desired outputs. In other words, our training process requires ground truth example data for the correct pop, push, and shift behavior in addition to the desired outputs. However, our model is able to generalize the behavior beyond example data as we will see later in the experiments.

If we want to simulate the behavior of a known LR(1)-automaton, the training data is simple to generate: We merely have to execute Algorithm 1 and record the rules that are applied, which directly yield the desired pop and push actions. The desired shift action is constant 1 and the desired output is 1 whenever the stack is  $\mathcal{S} = A$  for an accepting nonterminal  $A \in F$  and 0 otherwise. Importantly, we only need to know the behavior of the automaton on short training examples to learn the general stack interaction from demonstration.

If we do not have an LR(1)-automaton available to generate the training data, we require some other heuristic, which is dependent on the dataset. In our copy experiments, for example, we use pop and push actions to normalize the length of the stack at crucial points of the computation, thus making memory recall much simpler compared to a regular recurrent neural network.

Once training data in form of these tuples is constructed, we apply Algorithm 4 to re-order this training data into input-output pairs for each of the functions  $c^{\text{pop}}$ ,  $c^{\text{push}}$ ,  $c^{\text{shift}}$ , and  $c^{\text{out}}$ . This algorithm is a variation of Algorithm 2, controlled via teacher forcing. Finally, once these input-output-pairs are collected, we can train  $c^{\text{pop}}$ ,  $c^{\text{push}}$ ,  $c^{\text{shift}}$ , and  $c^{\text{out}}$  via any classification or regression scheme. In this paper, we opt for classic radial basis function support vector machines (RBF-SVMs)<sup>1</sup> with automatically chosen kernel width as implemented in scikit-

<sup>1</sup>The only exception is  $c^{\text{out}}$  in the copy tasks, where a linear regression is sufficient.

---

**Algorithm 4** An algorithm to re-order training data from a 5-tuple  $(\bar{x}, \mathbf{J}, \mathbf{A}, \vec{\rho}, \mathbf{Y})$  of inputs  $\bar{x} \in \Sigma^T$ , desired pop actions  $\mathbf{J} \in \{0, \dots, J\}^{T+1 \times M}$ , desired push actions  $\mathbf{A} \in \Phi^{T+1 \times M}$ , desired shift actions  $\vec{\rho} \in \{0, 1\}^{T+1}$ , and desired outputs  $\mathbf{Y} \in \mathbb{R}^{T+1 \times L}$  into input-output pairs for training the functions  $c^{\text{pop}}$ ,  $c^{\text{push}}$ ,  $c^{\text{shift}}$ , and  $c^{\text{out}}$ . We assume that a reservoir  $(\mathbf{U}, \mathbf{W}, \sigma)$  with  $\mathbf{U} \in \mathbb{R}^{m \times (n+K)}$ ,  $\mathbf{W} \in \mathbb{R}^{m \times m}$ , and  $\sigma: \mathbb{R} \rightarrow \mathbb{R}$ .

---

```

1: function RSM-TRAIN(Input training data  $(\bar{x}, \mathbf{J}, \mathbf{A}, \vec{\rho}, \mathbf{Y})$ )
2:   Initialize  $H^{\text{pop/push}}$ ,  $Y^{\text{pop}}$ , and  $Y^{\text{push}}$  as empty lists.
3:   Initialize an empty stack  $\mathcal{S}$ . Initialize  $\vec{h}_0 \leftarrow \vec{0}$  and  $\vec{g}_1 \leftarrow \vec{0}$ .
4:   for  $t \leftarrow \{1, \dots, T+1\}$  do
5:      $\vec{h}_t \leftarrow \sigma(\mathbf{U} \cdot \bar{x}_t + \mathbf{W} \cdot \vec{h}_{t-1})$ .
6:     for  $\tau \leftarrow \{1, \dots, M\}$  do
7:       Append  $(\vec{h}_t, \vec{g}_t)$  to  $H^{\text{pop/push}}$ .
8:       Append  $j_{t,\tau}$  to  $Y^{\text{pop}}$  and  $\vec{a}_{t,\tau}$  to  $Y^{\text{push}}$ .
9:       Pop  $j_{t,\tau}$  elements from  $\mathcal{S}$ .
10:      if  $\vec{a}_{t,\tau} \neq \vec{0}$  then
11:        Push  $\vec{a}_{t,\tau}$  onto  $\mathcal{S}$ .
12:      end if
13:       $\vec{g}_t \leftarrow \vec{0}$ .
14:      for  $\vec{s} \leftarrow$  elements of  $\mathcal{S}$  do
15:         $\vec{g}_t \leftarrow \sigma(\mathbf{U} \cdot \vec{s} + \mathbf{W} \cdot \vec{g}_t)$ .
16:      end for
17:      if  $j_{t,\tau} = 0$  and  $\vec{a}_{t,\tau} = \vec{0}$  then
18:        Break.
19:      end if
20:    end for
21:    if  $\rho_t > 0$  then
22:      Push  $\bar{x}_t$  onto  $\mathcal{S}$ .
23:       $\vec{g}_{t+1} \leftarrow \sigma(\mathbf{U} \cdot \bar{x}_t + \mathbf{W} \cdot \vec{g}_t)$ .
24:    else
25:       $\vec{g}_{t+1} \leftarrow \vec{g}_t$ .
26:    end if
27:  end for
28:  Convert  $H^{\text{pop/push}}$ ,  $Y^{\text{pop}}$ , and  $Y^{\text{push}}$  to matrices.
29:  Set up a  $T+1 \times 2m$  matrix  $\mathbf{H}$  with rows  $(\vec{h}_t, \vec{g}_t)$ .
30:  return  $(H^{\text{pop/push}}, Y^{\text{pop}})$ ,  $(H^{\text{pop/push}}, Y^{\text{push}})$ ,  $(\mathbf{H}, \vec{\rho})$ ,  $(\mathbf{H}, \mathbf{Y})$ .
31: end function

```

---

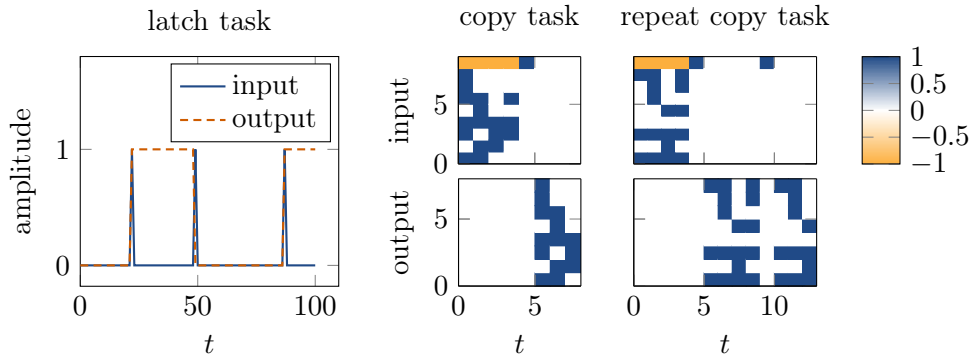


Figure 4: Example input and output sequences for the three NTM benchmark data sets.

learn (Pedregosa et al., 2011). Because the objective of RBF-SVMs is convex, the training is fast and globally optimal.

## 4 Experiments

We evaluate reservoir stack machines (RSMs) on three benchmark tasks for Neural Turing Machines (NTMs) (Collier and Beel, 2018) as well as six context-free languages. In more detail, the three NTM benchmark tasks are:

**latch:** Both input and output are binary and one-dimensional and the output should jump from zero to one and back whenever there is a one on the input. Equivalently, this task can be expressed via the regular expression  $(0^*10^*)^*0^*10^*$ , i.e. we accept any word that contains an odd number of ones. We sample training words up to length 50 and test words starting from length 50 from a probabilistic context-free grammar<sup>2</sup>.

**copy:** The input is a 1-20 time step long sequence of 8 random bits, followed by a special end-of-sequence token on an extra channel (this channel is -1 during the input sequence). The output should be zero until this extra one, after which the output should be a copy of the input signal.

**repeat copy:** The same as copy but where the input sequence has length up to 10 and the end-of-sequence token can occur up to 10 times (up to 20 times in the testing data). After each token, the input sequence should be copied again.

For a graphical illustration of the three datasets, refer to Figure 4. For the latch task, we use the LR(1)-rules  $(S0, \varepsilon, 2, S)$ ,  $(S1, \varepsilon, 2, A)$ ,  $(A0, \varepsilon, 2, A)$ ,  $(A1, \varepsilon, 2, S)$ ,  $(0, \varepsilon, 1, A)$  with the accepting nonterminal  $S$  to generate the desired stack behavior for the training data. For the copy task, we use the following heuristic: We always shift the input onto the stack, but we do not use pop or push until the end-of-sequence token appears. At that point, we fill up the stack with a placeholder nonterminal  $S$  until the stack has length 20. Then, we continue. By this construction, the output can be constructed by simply copying the 20th stack element to the output, i.e. the reservoir must be rich enough such that the 20th stack element can be reconstructed via  $c^{\text{out}}$  from  $\vec{g}_t$ . For this purpose, we use linear ridge regression as implemented in sklearn (with the  $\alpha$  hyperparameter being part of the hyperparameter optimization) because a linear operator is provably sufficient to reconstruct past inputs, at

<sup>2</sup>For the precise parameters and experimental details, refer to [https://gitlab.com/bpaassen/reservoir\\_stack\\_machines](https://gitlab.com/bpaassen/reservoir_stack_machines).



Dyck1	Dyck2	Dyck3	$a^n b^n$	Palindromes	JSON
$(SS, \varepsilon, 2, S)$	$(SS, \varepsilon, 2, S)$	$(SS, \varepsilon, 2, S)$	$(aSb, \varepsilon, 3, S)$	$(aSa, \varepsilon, 3, S)$	$(\{\}, \varepsilon, 2, V)$
$((S), \varepsilon, 3, S)$	$((S), \varepsilon, 3, S)$	$((S), \varepsilon, 3, S)$	$(a, b, \varepsilon, S)$	$(bSb, \varepsilon, 3, S)$	$([], \varepsilon, 2, V)$
	$([S], \varepsilon, 3, S)$	$([S], \varepsilon, 3, S)$		$(\$, \varepsilon, 1, S)$	$(\{O\}, \varepsilon, 3, V)$
		$(\{S\}, \varepsilon, 3, S)$			$([A], \varepsilon, 3, V)$
$((, ), 0, S)$	$((, ), 0, S)$	$((, ), 0, S)$			$(n, \varepsilon, 1, V)$
	$([, ], 0, S)$	$([, ], 0, S)$			$(s, \varepsilon, 1, V)$
		$(\{, \}, 0, S)$			$(k : V, O, \varepsilon, 5, O)$
					$(k : V, }, 3, O)$
					$(V, A, \varepsilon, 3, A)$
					$(V, ], 1, A)$
$()()$	$()[]$	$()\{\}$	$aaabbb$	$ab\$ba$	$\{k : [n, n], k : s\}$

Table 1: The list of rules for the LR(1)-automata of all six language data sets and an example word from each language. Terminal symbols are colored orange, nonterminals blue.

least for Legendre reservoirs (Voelker et al., 2019). For repeat copy, we use the same scheme, except for one difference: If a second end-of-sequence token appears, we pop elements from the stack until no end-of-sequence token is on the stack anymore and then continue as before.

Our six language data sets are:

**Dyck1/2/3:** Deterministic context-free languages of balanced bracket pairs with one, two, and three different kinds of brackets respectively, as suggested by Suzgun et al. (2019).

$a^n b^n$ : The language  $\mathcal{L}_{a^n b^n}$  from Equation 3.

**Palindrome:** The language of palindromes over the letters  $a$  and  $b$  with a  $\$$  symbol in the center from Equation 4.

**JSON:** A simplified version of the javascript object notation (JSON; <https://www.json.org>), where we represent numbers with the symbol  $n$ , strings with the symbol  $s$ , and keys with the symbol  $k$ .

The rules of the LR(1)-automata for all six languages are shown in Table 1, including an example word from each language. The accepting nonterminals of the automata are  $\{S\}$  for the first five and  $\{V\}$  for the last automaton. It is easy to show that none of these languages is Chomsky-3 by using the pumping lemma (Rabin and Scott, 1959; Sperschneider and Hammer, 1996). For each language task, our desired output is  $y_t = 1$  if the word up to  $t$  is in the language and  $y_t = 0$  otherwise.

We use the ground-truth LR(1)-automaton to annotate the training data with the desired stack behavior. However, we sample the training words up to length 50 and the evaluation words starting from length 50 as suggested by Suzgun et al. (2019), thus demonstrating that our model can generalize beyond the shown examples. We sample the words from a probabilistic context-free grammar for the respective language<sup>3</sup>.

In all experiments, we sample 100 random sequences for training and another 100 sequences for testing. To obtain statistics, we perform 10 repeats of every experiment. For hyperparameter optimization, we sample 20 random sets of hyperparameters<sup>3</sup>, each evaluated on 3 separate datasets of 100 training sequences and 100 test sequences.

We evaluate three kinds of reservoir for the RSM, namely random Gaussian numbers (rand), a cycle reservoir with jumps (CRJ) (Rodan and Tiño, 2012), and a Legendre delay

<sup>3</sup>For the precise parameters and experimental details, refer to [https://gitlab.com/bpaassen/reservoir\\_stack\\_machines](https://gitlab.com/bpaassen/reservoir_stack_machines).

Table 2: The mean absolute error ( $\pm$  std.) on the test data for all models for the NTM benchmark datasets. All results with mean and standard deviation below  $10^{-2}$  are bold-faced.

model	latch	copy	repeat copy
rand-ESN	$0.43 \pm 0.02$	$0.23 \pm 0.00$	$0.36 \pm 0.01$
CRJ-ESN	$0.40 \pm 0.02$	$0.20 \pm 0.01$	$0.34 \pm 0.01$
LDN-ESN	$0.42 \pm 0.01$	$0.22 \pm 0.00$	$0.30 \pm 0.01$
GRU	<b><math>0.00 \pm 0.00</math></b>	$0.22 \pm 0.00$	$0.32 \pm 0.01$
SRNN	$0.30 \pm 0.03$	$0.21 \pm 0.01$	$0.36 \pm 0.01$
DSM	<b><math>0.00 \pm 0.00</math></b>	$0.23 \pm 0.01$	$0.33 \pm 0.02$
rand-RSM	<b><math>0.00 \pm 0.00</math></b>	$0.36 \pm 0.03$	$0.40 \pm 0.01$
CRJ-RSM	$0.35 \pm 0.02$	$0.22 \pm 0.00$	$0.08 \pm 0.01$
LDN-RSM	<b><math>0.00 \pm 0.00</math></b>	<b><math>0.00 \pm 0.00</math></b>	<b><math>0.00 \pm 0.00</math></b>

network (LDN) (Voelker et al., 2019). As baseline, we compare against a standard echo state network (Jaeger and Haas, 2004), which attempts to predict the desired output via linear regression from the reservoir state  $\vec{h}_t$ . We expect that these networks should fail because they do not have the computational power to recognize languages that require arbitrarily long memory (Hammer and Tiño, 2003). For all reservoirs, we used 256 neurons to maintain equal representational power between models<sup>4</sup>.

We also include three baselines from deep learning, namely a gated recurrent unit (GRU) (Cho et al., 2014), the stack recurrent neural network (SRNN) (Suzgun et al., 2019), and a deep version of our reservoir stack machine (DSM), where we replace the reservoir with a GRU. For GRUs, we used the implementation provided by pyTorch (Paszke et al., 2019) and for SRNNs the reference implementation of Suzgun et al.<sup>5</sup>. We trained all deep models with Adam (Kingma and Ba, 2015) with a learning rate of  $10^{-3}$ , weight decay of  $10^{-8}$ , and 10,000 epochs, where we processed a single word in each epoch. As for the reservoir models, we used 256 neurons for each model in a single recurrent layer.

Note that we do *not* compare against reservoir memory machines (RMMs) (Paaßen et al., 2020) because there is no strategy to generate training data for the LR(1) language datasets (refer to Theorem 2). For the latch, copy, and repeat copy datasets, it has already been shown that RMMs achieve zero error (Paaßen et al., 2020).

We performed all experiments on a consumer-grade 2017 laptop with Intel i7 CPU.

Tables 2 and Table 3 report the mean average error across data sets and models. We observe that the reservoir stack machine with Lagrange delay reservoir (LDN-RSM) achieves almost zero error (bold-faced) on all datasets. While other reservoirs succeed on the language tasks, they fail on the copy and repeat copy task. This is to be expected because these tasks require a lossless reconstruction of past states from the stack, which the Lagrange delay network is designed for (Voelker et al., 2019).

Echo state networks without a stack fail almost all tasks, except for the  $a^n b^n$ , palindrome, and JSON languages. This corresponds to the theoretical findings of Hammer and Tiño (2003), indicating that ESNs cannot recognize general regular languages, let alone deter-

<sup>4</sup>We observed that 256 neurons were insufficient for the copy task. In this case, we increased the number of neurons to 512. Further, we report reference results for ESNs with 1024 neurons in the appendix.

<sup>5</sup>[https://github.com/suzgunmirac/marnns/blob/master/models/rnn\\_models.py](https://github.com/suzgunmirac/marnns/blob/master/models/rnn_models.py)

Table 3: The mean absolute error ( $\pm$  std.) on the test data for all models for the language datasets. All results with mean and standard deviation below  $10^{-2}$  are bold-faced.

model	Dyck1	Dyck2	Dyck3	$a^n b^n$	Palindrome	JSON
rand-ESN	0.13 $\pm$ 0.02	0.23 $\pm$ 0.05	0.28 $\pm$ 0.06	<b>0.00 <math>\pm</math> 0.00</b>	<b>0.00 <math>\pm</math> 0.00</b>	<b>0.01 <math>\pm</math> 0.00</b>
CRJ-ESN	0.11 $\pm$ 0.01	0.12 $\pm$ 0.02	0.14 $\pm$ 0.03	<b>0.00 <math>\pm</math> 0.00</b>	<b>0.01 <math>\pm</math> 0.00</b>	0.01 $\pm$ 0.00
LDN-ESN	0.17 $\pm$ 0.02	0.23 $\pm$ 0.02	0.26 $\pm$ 0.03	<b>0.00 <math>\pm</math> 0.00</b>	0.07 $\pm$ 0.01	0.10 $\pm$ 0.01
GRU	0.04 $\pm$ 0.01	0.05 $\pm$ 0.01	0.06 $\pm$ 0.01	<b>0.00 <math>\pm</math> 0.00</b>	<b>0.00 <math>\pm</math> 0.00</b>	<b>0.00 <math>\pm</math> 0.00</b>
SRNN	0.02 $\pm$ 0.03	<b>0.00 <math>\pm</math> 0.00</b>	0.03 $\pm$ 0.03	<b>0.00 <math>\pm</math> 0.00</b>	0.05 $\pm$ 0.14	0.08 $\pm$ 0.16
DSM	<b>0.00 <math>\pm</math> 0.00</b>	<b>0.00 <math>\pm</math> 0.00</b>	<b>0.00 <math>\pm</math> 0.00</b>	<b>0.00 <math>\pm</math> 0.00</b>	<b>0.00 <math>\pm</math> 0.00</b>	<b>0.00 <math>\pm</math> 0.00</b>
rand-RSM	<b>0.00 <math>\pm</math> 0.00</b>	<b>0.00 <math>\pm</math> 0.00</b>	<b>0.00 <math>\pm</math> 0.00</b>	<b>0.00 <math>\pm</math> 0.00</b>	<b>0.00 <math>\pm</math> 0.00</b>	<b>0.00 <math>\pm</math> 0.00</b>
CRJ-RSM	<b>0.00 <math>\pm</math> 0.00</b>	<b>0.00 <math>\pm</math> 0.00</b>	<b>0.00 <math>\pm</math> 0.00</b>	<b>0.00 <math>\pm</math> 0.00</b>	<b>0.00 <math>\pm</math> 0.00</b>	<b>0.00 <math>\pm</math> 0.00</b>
LDN-RSM	<b>0.00 <math>\pm</math> 0.00</b>	<b>0.00 <math>\pm</math> 0.00</b>	<b>0.00 <math>\pm</math> 0.00</b>	<b>0.00 <math>\pm</math> 0.00</b>	<b>0.00 <math>\pm</math> 0.00</b>	<b>0.00 <math>\pm</math> 0.00</b>

Table 4: The mean runtime ( $\pm$  std.) in seconds as measured by Python’s `time` function for all models on the NTM benchmark datasets.

model	latch	copy	repeat copy
rand-ESN	1.17 $\pm$ 0.12	0.48 $\pm$ 0.01	0.32 $\pm$ 0.01
CRJ-ESN	0.76 $\pm$ 0.11	0.11 $\pm$ 0.01	0.18 $\pm$ 0.01
LDN-ESN	1.71 $\pm$ 0.19	0.19 $\pm$ 0.01	0.23 $\pm$ 0.01
GRU	97.1 $\pm$ 4.1	108.3 $\pm$ 12.2	191.5 $\pm$ 74.4
SRNN	295.0 $\pm$ 12.4	314.5 $\pm$ 24.4	536.3 $\pm$ 113.5
DSM	619.3 $\pm$ 20.3	548.9 $\pm$ 30.3	1045.4 $\pm$ 211.6
rand-RSM	10.25 $\pm$ 0.62	5.20 $\pm$ 0.14	9.61 $\pm$ 0.65
CRJ-RSM	51.27 $\pm$ 5.16	4.11 $\pm$ 0.14	7.78 $\pm$ 0.38
LDN-RSM	26.74 $\pm$ 2.04	3.36 $\pm$ 0.13	10.25 $\pm$ 0.46

ministic context-free languages. The fact that some languages can still be solved is likely due to the fact that all test sequences stay within the memory capacity of our model.

Interestingly, the deep models also fail for many tasks, especially the copy and repeat copy task. This corresponds to prior findings of Collier and Beel (2018) that these tasks likely require tens of thousands of unique sequences to be learned in a memory-augmented neural network, whereas we only present 100 distinct sequences. Even for the language tasks, Stack-RNNs often fail, indicating that the correct stack behavior is not easy to learn. Indeed, even the deep variation of our model (DSM), which receives the same amount of training data as the reservoir stack machine, fails on the copy and repeat copy task, indicating that these tasks are not trivial to learn.

Tables 4 and 5 show the runtime needed for training and evaluating all models on all datasets. Unsurprisingly, RSMs take more time compared to ESNs due to the stack mechanism and because we need to train four classifiers instead of one linear regression, yielding a factor of  $\approx 10$  for LDN-RSMs. GRUs are about seven times slower compared to an LDN-RSM, SRNNs are about 20 times slower, and DSMs about 35 times slower.

Overall, we conclude that reservoir stack machines can solve tasks that are impossible to solve for ESNs and hard to solve for deep networks. Additionally, while RSMs are much slower compared to pure echo state networks, they are still much faster compared to deep networks (even for small training data sets).

Table 5: The mean runtime ( $\pm$  std.) in seconds as measured by Python’s `time` function for all models on the language datasets.

model	Dyck1	Dyck2	Dyck3	$a^n b^n$	Palindrome	JSON
rand-ESN	$1.68 \pm 0.30$	$1.78 \pm 0.13$	$1.93 \pm 0.40$	$0.89 \pm 0.04$	$1.29 \pm 0.29$	$1.01 \pm 0.06$
CRJ-ESN	$1.27 \pm 0.32$	$0.88 \pm 0.13$	$0.98 \pm 0.21$	$0.38 \pm 0.02$	$0.61 \pm 0.16$	$0.80 \pm 0.05$
LDN-ESN	$2.27 \pm 0.29$	$1.28 \pm 0.10$	$1.22 \pm 0.26$	$0.92 \pm 0.12$	$1.04 \pm 0.32$	$0.95 \pm 0.15$
GRU	$45.6 \pm 3.1$	$45.5 \pm 3.0$	$42.3 \pm 2.0$	$52.6 \pm 8.9$	$47.9 \pm 2.5$	$37.8 \pm 1.8$
SRNN	$114.8 \pm 10.7$	$114.0 \pm 10.3$	$102.5 \pm 6.2$	$131.0 \pm 24.7$	$122.0 \pm 9.5$	$88.0 \pm 5.9$
DSM	$285.0 \pm 31.3$	$282.1 \pm 30.8$	$249.1 \pm 17.7$	$270.3 \pm 37.5$	$290.3 \pm 27.0$	$217.6 \pm 17.3$
rand-RSM	$11.52 \pm 0.72$	$16.64 \pm 1.40$	$17.80 \pm 1.78$	$4.01 \pm 0.08$	$6.09 \pm 0.42$	$13.16 \pm 1.18$
CRJ-RSM	$11.08 \pm 0.69$	$15.06 \pm 1.14$	$17.08 \pm 1.61$	$4.15 \pm 0.07$	$7.06 \pm 0.50$	$11.42 \pm 0.99$
LDN-RSM	$13.71 \pm 0.97$	$14.34 \pm 1.40$	$14.22 \pm 0.84$	$5.06 \pm 0.22$	$6.49 \pm 0.42$	$11.90 \pm 0.79$

## 5 Conclusion

In this paper, we presented the reservoir stack machine (RSM), a combination of an echo state network with a stack. We have shown that a sufficiently rich reservoir suffices to simulate any LR(1)-automaton, whereas a constant-sized memory only suffices for finite state automata. We have evaluated our model on three benchmark tasks for Neural Turing Machines (latch, copy, and repeat copy) and six deterministic context-free languages. ESNs struggled with all three benchmark tasks and most LR(1) languages and even deep models were unable to solve the copy and repeat copy task.

By contrast, RSMS could solve all tasks with zero generalization error. For the LR(1) languages, this is independent of the choice of reservoir, whereas a Legendre delay network was required for the copy and repeat copy task. The Legendre network has the advantage that it can provably decode past inputs via a linear operator, which simplifies the output function on these tasks.

We admit that a crucial limitation of RSMS is that they require additional training data in the form of desired pop, push, and shift behavior. However, RSMS can generalize this behavior from examples to longer sequences, indicating that actual learning takes place. Further, even with this additional training data, a deep learning model failed to solve tasks that an LDN-RSM could solve, indicating that the learning task is still non-trivial. Accordingly, we conclude that RSMS provide a novel way to learn difficult stack behavior within seconds and with few short reference sequences.

Future research could extend the reservoir stack machine with a second stack to a full Turing machine and try to find mechanisms in order to construct desired pop, push, and shift behavior for training data. Even in the present form, though, we hope that the reservoir stack machine provides an interesting new avenue to explore memory-augmented neural networks in a way that is faster and more reliable to train.

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Table 6: The mean absolute error ( $\pm$  std.) on the test data for ESN models with 1024 neurons. All results with mean and standard deviation below  $10^{-2}$  are bold-faced.

dataset	rand-ESN	CRJ-ESN	LDN-ESN
latch	$0.42 \pm 0.03$	$0.39 \pm 0.01$	$0.43 \pm 0.01$
copy	$0.24 \pm 0.01$	$0.21 \pm 0.00$	$0.22 \pm 0.00$
repeat copy	$0.53 \pm 0.04$	$0.33 \pm 0.01$	$0.31 \pm 0.01$
Dyck1	$0.14 \pm 0.02$	$0.14 \pm 0.02$	$0.18 \pm 0.02$
Dyck2	$0.58 \pm 0.64$	$0.15 \pm 0.02$	$0.23 \pm 0.02$
Dyck3	$0.34 \pm 0.08$	$0.16 \pm 0.03$	$0.27 \pm 0.03$
$a^n b^n$	<b><math>0.00 \pm 0.00</math></b>	<b><math>0.00 \pm 0.00</math></b>	<b><math>0.00 \pm 0.00</math></b>
Palindrome	<b><math>0.00 \pm 0.00</math></b>	<b><math>0.00 \pm 0.00</math></b>	$0.07 \pm 0.01$
JSON	<b><math>0.00 \pm 0.00</math></b>	<b><math>0.00 \pm 0.00</math></b>	$0.10 \pm 0.01$

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## A Additional Experiments

To verify that the worse results for pure echo state networks are not just due to less parameters, we repeated all experiments with a reservoir with 1024 neurons, i.e. four times the number as for the RSM models to counteract the four output functions that the RSM model has available. The errors are shown in Table 6. Qualitatively, we observe the same results as in Tables 2 and 3, namely that the ESN models are unable to solve latch, copy, or repeat copy, and none of the Dyck languages, but that some reservoirs achieve zero error on  $a^n b^n$ , palindromes, and JSON.

The runtimes are shown in Table 7. As to be expected, we observe longer runtimes than in Tables 4 and 5 because the reservoirs are much larger. As before, the CRJ is the fastest reservoir overall and scales roughly linearly with the reservoir size, whereas the random and Legendre reservoirs appear to scale worse than linear, approaching and sometimes exceeding the runtime of the reservoir stack machine models with 256 neurons.



Table 7: The mean runtime ( $\pm$  std.) in seconds as measured by Python’s `time` function for ESN models with 1024 neurons.

dataset	rand-ESN	CRJ-ESN	LDN-ESN
latch	$6.00 \pm 0.22$	$1.43 \pm 0.06$	$12.85 \pm 0.44$
copy	$0.93 \pm 0.04$	$0.20 \pm 0.03$	$0.38 \pm 0.03$
repeat copy	$20.88 \pm 1.35$	$1.61 \pm 0.22$	$3.83 \pm 0.18$
Dyck1	$15.34 \pm 0.67$	$1.71 \pm 0.10$	$13.33 \pm 0.84$
Dyck2	$15.66 \pm 0.53$	$1.81 \pm 0.15$	$8.15 \pm 0.41$
Dyck3	$9.28 \pm 0.55$	$1.76 \pm 0.11$	$6.29 \pm 0.54$
$a^n b^n$	$10.54 \pm 0.10$	$0.89 \pm 0.01$	$8.73 \pm 0.08$
Palindrome	$11.28 \pm 0.82$	$1.15 \pm 0.31$	$6.48 \pm 0.42$
JSON	$6.59 \pm 0.63$	$1.78 \pm 0.26$	$4.03 \pm 0.26$