Generic Modal Cut Elimination Applied to Conditional Logics

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Abstract. We develop a general criterion for cut elimination in sequent calculi for propositional modal logics, which rests on absorption of cut, contraction, weakening and inversion by the purely modal part of the rule system. Our criterion applies also to a wide variety of logics outside the realm of normal modal logic. We give extensive example instantiations of our framework to various conditional logics. For these, we obtain fully internalised calculi which are substantially simpler than those known in the literature, along with leaner proofs of cut elimination and complexity. In one case, conditional logic with modus ponens and conditional excluded middle, cut elimination and complexity are explicitly stated as open in the literature.

1 Introduction

Cut elimination, originally invented by Gentzen [5], is one of the core concepts of proof theory and plays a major role in particular for algorithmic aspects of logic, including the subformula property, the complexity of automated reasoning and, via interpolation, modularity issues. The large number of logical calculi that are currently in use, in particular in various areas of computer science, motivates efforts to define families of sequent calculi that cover a variety of logics and admit uniform proofs of cut elimination, enabled by suitable sufficient conditions. Here, we present such a method for modal sequent calculi that in particular applies also to non-normal modal logics, which appear e.g. in concurrency and especially in knowledge representation. We use a separation of the modal calculi into a fixed underlying propositional part and a modal part; the core of our criterion is absorption of cut by the modal rules. This concept generalises the notion of resolution closed rule set [9, 12], dropping the assumption that the logic at hand is rank-1, i.e. axiomatised by formulas in which the nesting depth of modal operators is uniformly equal to 1 (such as K).

Our method is reasonably simple and intuitive, and nevertheless applies to a wide range of modal logics. While we use normal modal logics such as K and T as running examples to illustrate our concepts at the time of introduction, our main example applications are conditional logics, which have a binary modal operator read as a non-monotonic implication (unlike default logics, conditional logics allow nested non-monotonic implications). In particular, we prove cut-elimination (hence, since the generic systems under consideration are analytic, the subformula property) for the conditional logics CK, CKMP, CKCEM, and CKMPCEM using our generic procedure. An easy analysis of proof search in the arising cut-free calculi moreover establishes that the satisfiability problem of each of these logics is in *PSPACE*, but this bound is only tight for CK and CKMP whereas the provability problem in extensions of CKCEM can be solved in *coNP*. We point out that while (different) cut-free labelled sequent calculi for CK, CKMP, CKCEM, and some further conditional logics, as well as the ensuing upper complexity bounds, have previously been presented by Olivetti et al., the corresponding issues for CKMPCEM have explicitly been left as open problems [8].

Related work A set of sufficient conditions for a sequent calculus to admit cut elimination and a subsequent analysis of the complexity *of cut elimination* (not proof search) is presented in [10]. The range of application of this method is very wide and encompasses e.g. first-order logic, the modal logic $S4$, linear logic, and intuitionistic propositional logic. This generality is reflected in the fact that the method as a whole is substantially more involved than ours. A simpler method for a different and comparatively restrictive class of calculi, so-called canonical calculi, is considered in [1]; this method does not apply to typical modal systems, as it considers only so-called *canonical* rules, i.e., left and right introduction rules for connectives which permit adding a common context simultaneously in the premise and the conclusion. (In fact, it might be regarded as the essence of modal logic that its rules fail to be canonical; e.g. the necessitation rule $A/\Box A$ does not generalise to $\Gamma, A/\Gamma, \Box A$ for a sequent Γ.) Moreover, the format of the rules in *op.cit.* does not allow for the introduction of more than one occurrence of a logical connective, which is necessary even for the most basic modal logics. The same applies to [4]. In [3], logical rules are treated on an individual basis, which precludes the treatment of cuts between two rule conclusions. Overall, our notion of absorption is substantially more general when compared to similar notions in the papers discussed above, which stipulate that cuts between left and right rules for the same connective are absorbed by structural rules. In our own earlier work [9], we have considered a special case of the method presented here in the restricted context of *rank-1* logics; in particular, these results did not cover logics such as $K4$, CKMP, or CKMPCEM.

2 Preliminaries and Notation

A *modal similarity type* (or modal signature) is a set Λ of modal operators with associated arities that we keep fixed throughout the paper. Given a set V of propositional variables, the set $\mathcal{F}(\Lambda)$ of Λ-formulas is given by the grammar

$$\mathcal{F}(\Lambda) \ni A, B ::= \bot \mid p \mid \neg A \mid A \wedge B \mid \heartsuit(A_1, \dots, A_n)$$

where $p \in V$ and $\heartsuit \in \Lambda$ is n-ary. We use standard abbreviations of the other propositional connectives $\top$, $\vee$ and $\rightarrow$. A *Λ-sequent* is a finite multiset of Λ-formulas, and the set of Λ-sequents is denoted by $\mathsf{S}(\Lambda)$. We write the multiset union of Γ and Δ as Γ, Δ and identify a formula $A \in \mathcal{F}(\Lambda)$ with the singleton sequent containing only A. If $S \subseteq \mathcal{F}(\Lambda)$ is a set of formulas, then an S-*substitution* is a mapping $\sigma : V \to S$. We denote the result of uniformly substituting $\sigma(p)$ for p in a formula A by $A\sigma$. This extends pointwise to Λ-sequents so that $\Gamma\sigma = A_1\sigma, \dots, A_n\sigma$ if $\Gamma = A_1, \dots, A_n$. If $S \subseteq \mathcal{F}(\Lambda)$ is a set of Λ-formulas and $A \in \mathcal{F}(\Lambda)$, we say that A is a *propositional consequence* of S if there exist $A_1, \dots, A_n \in S$ such that $A_1 \wedge \dots \wedge A_n \to A$ is a substitution instance of a propositional tautology. We write $S \vdash_{\mathsf{PL}} A$ if A is a propositional consequence of S and $A \vdash_{\mathsf{PL}} B$ for $\{A\} \vdash_{\mathsf{PL}} B$ for the case of single formulas.

3 Modal Deduction Systems

To facilitate the task of comparing the notion of provability in both Hilbert and Gentzen type proof systems, we introduce the following notion of a proof rule that can be used, without any modifications, in both systems.

Definition 1. *A* Λ-rule *is of the form* $\frac{\Gamma_1, \dots, \Gamma_n}{\Gamma_0}$ *where* $n \geq 0$ *and* $\Gamma_0, \dots, \Gamma_n$ *are* Λ*-sequents. The sequents* $\Gamma_1, \dots, \Gamma_n$ *are the* premises *of the rule and* Γ_0 *its* conclusion. *A rule* $\frac{}{\Gamma_0}$ *without premises is called a* Λ-axiom*, which we denote by just its conclusion,* Γ_0*. A* rule set *is just a set of* Λ*-rules, and we say that a rule set* R *is* substitution closed*, if* $\Gamma_1\sigma \dots \Gamma_n\sigma / \Gamma_0\sigma \in \mathsf{R}$ *whenever* $\Gamma_1 \dots \Gamma_n / \Gamma_0 \in \mathsf{R}$ *and* $\sigma : V \to \mathcal{F}(\Lambda)$ *is a substitution.*

In view of the sequent calculi that we introduce later, we read sequents disjunctively. Consequently, a rule $\Gamma_1, \dots, \Gamma_n / \Gamma_0$ can be used to prove the disjunction Γ_0, provided that $\bigvee \Gamma_i$ is provable, for all $1 \leq i \leq n$. We emphasise that a rule is an expression of the object language, i.e. it does not contain meta-linguistic variables. As such, it represents a specific deduction step rather than a family of possible deductions, which helps to economise on syntactic categories. In our examples, concrete rule sets are presented as instances of rule schemas.

Example 2. For the modal logics K, $K4$ and T, we fix the modal signature $\Lambda = \{\Box\}$ consisting of a single modal operator $\Box$ with arity one. The language of conditional logic is given by the similarity type $\Lambda = \{\Rightarrow\}$ where the conditional arrow $\Rightarrow$ has arity 2. We use infix notation and write $A \Rightarrow B$ instead of $\Rightarrow (A, B)$ for $A, B \in \mathcal{F}(\Lambda)$. Deduction over modal and conditional logics are governed by the following rule sets:

1. The rule set K associated to the modal logic K consists of all instances of the necessitation rule (N) and the distribution axiom (D) below.

   $$(\mathsf{N})\frac{A}{\Box A} \qquad (\mathsf{D})\Box(A \to B) \to (\Box A \to \Box B) \qquad (\mathsf{4})\Box\Box A \to \Box A \qquad (\mathsf{R})\Box A \to A$$

   The rule sets for T and $K4$ arise by extending this set with the reflexivity axiom (R) and the ($\mathsf{4}$)-axiom, respectively. We reserve the name (T) for the reflexivity rule in a cut-free system.

2. Conditional logic, e.g. the system CK of [2] is axiomatised by the rule set that consists of all instances of (RCEA) on the left, and (RCK) on the right below:

$$\frac{A \leftrightarrow A'}{(A \Rightarrow B) \leftrightarrow (A' \Rightarrow B)} \qquad \frac{B_1 \wedge \cdots \wedge B_n \rightarrow B}{(A \Rightarrow B_1) \wedge \cdots \wedge (A \Rightarrow B_n) \rightarrow (A \Rightarrow B)}$$

As additional axioms, we consider

$$(\mathsf{ID}) A \Rightarrow A \qquad (\mathsf{MP})(A \Rightarrow B) \rightarrow (A \rightarrow B) \qquad (\mathsf{CEM})(A \Rightarrow B) \vee (A \Rightarrow \neg B)$$

that induce extensions of CK that we denote by juxtaposition of the respective axioms, e.g. $\mathsf{CKMPCEM}$ contains the rules for CK and the axioms (MP) and (CEM).

Rules with more than one premise arise through saturation of a given rule set under cut that e.g. leads to the rules (CK_g) and (MP_g) presented in Section 6.

The notion of deduction in modal Hilbert systems then takes the following form.

Definition 3. *Suppose* R *is a set of rules. The set of* R*-derivable formulas in the Hilbert-system given by* R *is the least set of formulas that*

- *contains* $A\sigma$ *whenever* A *is a propositional tautology and* σ *is a substitution*
- *contains* B *whenever it contains* A *and* $A \rightarrow B$
- *contains* $\bigvee \Gamma_0$ *whenever it contains* $\bigvee \Gamma_1, \ldots, \bigvee \Gamma_n$ *and* $\frac{\Gamma_1 \ldots \Gamma_n}{\Gamma_0} \in \mathsf{R}$.

We write $\mathsf{HR} \vdash A$ *if* A *is* R*-derivable.*

In other words, the set of derivable formulas is the least set that contains propositional tautologies, is closed under uniform substitution, modus ponens and application of rules. We will later consider Hilbert systems that induce the same provability predicate based on the following notion of admissibility.

Definition 4. *A rule set* R' *is* admissible *in* HR *if* $\mathsf{HR} \vdash A \iff \mathsf{H}(\mathsf{R} \cup \mathsf{R}') \vdash A$ *for all formulas* $A \in \mathcal{F}(\Lambda)$*. Two rule sets* R, R' *are* equivalent *if* R *is admissible in* HR' *and* R' *is admissible in* HR.

In words, R' is admissible in HR if adding the rules R' to those of R leaves the set of provable formulas unchanged. We note the following trivial, but useful consequence of admissibility.

Lemma 5. $\mathsf{HR} \vdash A$ *iff* $\mathsf{HR}' \vdash A$ *if* R *and* R' *are equivalent and* $A \in \mathcal{F}(\Lambda)$.

The next proposition is concerned with the structure of proofs in Hilbert systems and is the key for proving equivalence of Hilbert and Gentzen-type systems.

Proposition 6. *The set* $\mathsf{HT}(\mathsf{R}) = \{A \in \mathcal{F}(\Lambda) \mid \mathsf{HR} \vdash A\}$ *is the smallest set* S *of formulas that contains a formula* $A \in \mathcal{F}(\Lambda)$ *whenever there are rules* $\Theta_1/\Gamma_1, \ldots, \Theta_n/\Gamma_n \in \mathsf{R}$ *and substitutions* $\sigma_1, \ldots, \sigma_n : V \rightarrow \mathcal{F}(\Lambda)$ *such that* $\bigvee \Delta\sigma_i \in S$ *for all* $\Delta \in \Theta_i$ $(i = 1, \ldots, n)$ *and* $\{\bigvee \Gamma_1\sigma, \ldots, \bigvee \Gamma_n\sigma\} \vdash_{\mathsf{PL}} A$.

Proof. The inclusion $S \subseteq \mathsf{HT}(\mathsf{R})$ is immediate as $\mathsf{HT}(\mathsf{R})$ contains propositional tautologies, is closed under uniform substitution and modus ponens. For the reverse inclusion we show that S is closed under R-derivability as considered in Definition 3.

This is clear for all cases (propositional tautologies, uniform substitutions, rule application) except possibly modus ponens. So assume that $\mathsf{HR} \vdash A \to B$ and $\mathsf{HR} \vdash A$. By induction hypothesis, there are

- Rules $\Theta_1/\Gamma_1, \dots, \Theta_n/\Gamma_n$ and substitutions $\sigma_1, \dots, \sigma_n$ such that $\{\bigvee \Gamma_1\sigma_1, \dots, \bigvee \Gamma_n\sigma_n\} \vdash_{\mathsf{PL}} A \to B$
- Rules $\Sigma_1/\Delta_1, \dots, \Sigma_k/\Delta_k$ and substitutions $\tau_1, \dots, \tau_k$ such that $\{\bigvee \Delta_1\tau_1, \dots, \bigvee \Delta_k\tau_k\} \vdash_{\mathsf{PL}} A$

and moreover $\bigvee \Xi\sigma \in S$ whenever $\Xi \in \Theta_1, \dots, \Theta_n, \Sigma_1, \dots, \Sigma_k$. The claim follows, as $\{\Gamma_1\sigma_1, \dots, \Gamma_n\sigma_n, \Delta_1\tau_1, \dots, \Delta_k\tau_k\} \vdash_{\mathsf{PL}} B$.

In other words, in a modal Hilbert system, each provable formula is a propositional consequence of rule conclusions with provable premises. This result forms the basis of our comparison of Hilbert and Gentzen systems, and we show that cut elimination essentially amounts to the fact that – in the corresponding Hilbert system – each valid formula is a consequence of a *single* rule conclusion with provable premise.

We now set the stage for sequent systems that we are going to address in the remainder of the paper. The notion of derivability in the sequent calculus associated with a rule set R is formulated parametric in terms of a set X of additional rules that will later be instantiated with relativised versions of cut, weakening, contraction and inversion.

Definition 7. *Suppose R and X are sets of Λ-rules. The set of $\mathsf{RC}+\mathsf{X}$-derivable sequents in the Gentzen-system given by R is the least set of sequents that*

- *contains $A, \neg A, \Gamma$ for all sequents $\Gamma \in \mathsf{S}(\Lambda)$ and formulas $A \in \mathcal{F}(\Lambda)$*
- *contains $\neg\bot, \Gamma$ for all $\Gamma \in \mathsf{S}(\Lambda)$*
- *is closed under instances of the rule schemas*

$$\frac{\Gamma, \neg A, \neg B}{\Gamma, \neg(A \wedge B)} \qquad \frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \qquad \frac{\Gamma, A}{\Gamma, \neg\neg A}$$

  where $A \in \mathcal{F}(\Lambda)$ ranges over formulas and $\Gamma \subseteq \mathcal{F}(\Lambda)$ over multisets of formulas. We call the above rules the propositional rules *and the formula occurring in the conclusion but not in Γ* principal *in the respective rule.*
- *is closed under the rules in $\mathsf{R} \cup \mathsf{X}$, i.e. it contains Γ_0 whenever it contains $\Gamma_1, \dots, \Gamma_n$ for $\frac{\Gamma_1 \dots \Gamma_n}{\Gamma_0} \in \mathsf{R} \cup \mathsf{X}$.*

We write $\mathsf{GR} + \mathsf{X} \vdash \Gamma$ if Γ can be derived in this way and $\mathsf{GR} \vdash \Gamma$ if $\mathsf{X} = \emptyset$.

The set X of extra rules will later be instantiated with a relativised version of the cut rule and additional axioms that locally capture the effect of weakening, contraction and inversion, applied to rule premises. This allows to formulate *local* conditions for the admissibility of cut that can be checked on a per-rule basis.

Remark 8. Many other formulations of sequent systems only permit axioms of the form $\Gamma, p, \neg p$ where $p \in V$ is a propositional atom. The reason for being more liberal here is that this makes it easier to prove admissibility of uniform substitution. However, the price we have to pay is that inversion no longer preserves the height of the proof tree. For example, we have that $A \wedge B, \neg(A \wedge B)$ is provable with a proof tree of height one (being an axiom), but neither of $A, \neg(A \wedge B)$ and $B, \neg(A \wedge B)$ and $A \wedge B, \neg A, \neg B$ is are axioms, i.e provable with a proof tree of height 1.

The following proposition is readily established by an induction on the provability predicate $\mathsf{RH} \vdash$.

Proposition 9. *Suppose* $\Gamma \in \mathsf{S}(\Lambda)$ *is a sequent. Then* $\mathsf{RH} \vdash \bigvee \Gamma$ *if* $\mathsf{RG} \vdash \Gamma$.

The remainder of the paper is concerned with the converse of the above proposition, which relies on specific properties of the rule set R which we address next.

4 Generic Modal Cut Elimination

In order to establish the converse of Proposition 9 we need to establish that the cut rule is admissible in the Gentzen system GR defined by the ruleset R. Clearly, we cannot expect that cut elimination holds in general: it is well known (and easy to check) that the sequent system arising from the rule set consisting of all instances of (N) and (D), presented in Example 2 does *not* enjoy cut elimination. In other words, we have to look for constructions that allow us to transform a given rule set into one for which cut elimination holds. The main result of our analysis is that cut elimination holds if the rule set under consideration satisfies four crucial requirements that are local in the sense that they can be checked on a per-rule basis without the need of carrying out a fully-fledged cut-elimination proof: absorption of weakening, contraction, inversion and cut.

The first three properties can be checked for each rule individually and amount to the admissibility of the respective principle, and the last requirement amounts to the possibility of eliminating cut between a pair of rule conclusions. We emphasise that these properties can be checked locally for the modal rules, and cut elimination will follow automatically. It is not particularly surprising that cut elimination holds under these assumptions. However, isolating the four conditions above provides us with means to convert a modal Hilbert system into an equivalent cut-free sequent calculus. We now introduce relativised versions of the structural rules that will be the main tool in the proof of cut elimination.

Definition 10. *Suppose* Γ *is a* Λ*-sequent and let* $\mathsf{A}(\Gamma)$ *consist of the axioms*

- Γ, A *for all* $A \in \mathcal{F}(\Lambda)$
- Δ, A *if* $\Gamma = \Delta, A, A$ *for some* $\Delta \in \mathsf{S}(\Lambda), A \in \mathcal{F}(\Lambda)$
- Δ, A *if* $\Gamma = \Delta, \neg\neg A$ *for some* $\Delta \in \mathsf{S}(\Lambda), A \in \mathcal{F}(\Lambda)$
- $\Delta, \neg A_1, \neg A_2$ *if* $\Gamma = \Delta, \neg(A_1 \wedge A_2)$ *for some* $\Delta \in \mathsf{S}(\Lambda)$*,* $A_1, A_2 \in \mathcal{F}(\Lambda)$

– Δ, A_i *for* $i = 1, 2$ *if* $\Gamma = \Delta, (A_1 \wedge A_2)$ *for some* $\Delta \in \mathsf{S}(\Lambda)$, $A_1, A_2 \in \mathcal{F}(\Lambda)$

We say that a rule set R absorbs the structural rules *if*

$$\mathsf{GR} + \mathsf{A}(\Gamma_1) \cup \cdots \cup \mathsf{A}(\Gamma_n) \vdash \Gamma$$

for all $\frac{\Gamma_1 \ldots \Gamma_n}{\Gamma_0} \in \mathsf{R}$ *and all* $\Gamma \in \mathsf{A}(\Gamma_0)$.

In other words, a deduction step that applies weakening, contraction or inversion to a rule conclusion can be replaced by a (possibly different) rule where the corresponding structural rules are applied to the premises. We discuss a number of standard examples before stating that absorption of the structural rules implies their admissibility.

Example 11. The rule sets containing all instances of either of the following rule schema

$$(K)\frac{\neg A_1, \ldots, \neg A_n, A_0}{\neg\Box A_1, \ldots, \neg\Box A_n, \Box A_0, \Gamma} \qquad (T)\frac{\neg A, \neg\Box A, \Gamma}{\neg\Box A, \Gamma} \qquad (K4)\frac{\neg A_1, \neg\Box A_1, \ldots, \neg A_n, \neg\Box A_n, B}{\neg\Box A_1, \ldots, \neg\Box A_n, \Box B, \Gamma}$$

absorbs the structural rules. We note that (K) absorbs weakening due to the presence of Γ in the conclusion, and the absorption of contraction in (T) and $(K4)$ is a consequence of the presence of the negated $\Box$-formulas in the premise. The absorption of inversion in a consequence of the weakening context Γ in (K) and $(K4)$ and implied by duplicating the context Γ in (T). On the other hand, the rule sets defined by

$$\frac{\neg A_1, \ldots, \neg A_n, A_0}{\neg\Box A_1, \ldots, \neg\Box A_n, \Box A_0} \qquad \frac{\neg A, \Gamma}{\neg\Box A, \Gamma}$$

fail to absorb the structural rules: the rule on the left fails to absorb weakening, whereas the right-hand rule does not absorb contraction.

It should be intuitively clear that absorption of structural rules implies their admissibility, which we establish next.

Proposition 12. *Suppose* R *absorbs the structural rules. Then all instances of the rule schemas of weakening, contraction and inversion*

$$\frac{\Gamma}{\Gamma, A} \qquad \frac{\Gamma, A, A}{\Gamma, A} \qquad \frac{\Gamma, \neg\neg A}{\Gamma, A} \qquad \frac{\Gamma, \neg(A_1 \wedge A_2)}{\Gamma, \neg A_1, \neg A_2} \qquad \frac{\Gamma, A_1 \wedge A_2}{\Gamma, A_i}(i = 1, 2)$$

where $\Gamma \in \mathsf{S}(\Lambda)$ *and* $A, A_1, A_2 \in \mathcal{F}(\Lambda)$ *are admissible in* GR.

Proof. Standard induction on proofs in GR where the case of propositional rules is standard and the inductive case for modal rules immediately follows from absorption.

Remark 13. 1. The main purpose for introducing the notion of absorption of structural rules (Definition 10) is to have a handy criterion that guarantees admissibility of the structural rules (Proposition 12). Our definition offers a compromise between generality and simplicity. In essence, a rule set absorbs structural rules, if an application of weakening, contraction or inversion can be pushed up one level of the proof tree. A weaker version of Definition 10 would require that an application of weakening, contraction or inversion to a rule conclusion can be replaced by a sequence of deduction steps where the structural rule in question can not only be applied to the premises of the rule, but also freely anywhere else, provided that these additional applications are smaller in a well-founded ordering. However, we are presently not aware of any examples where this extra generality would be necessary.

2. In many Sequent systems, the statement of Proposition 12 can be strengthened to say that weakening, contraction and inversion are depth-preserving admissible, i.e. does not increase the height of the proof tree. This is in general false for the systems considered here as axioms are of the form $A, \neg A, \Gamma$ for $A \in \mathcal{F}(\Lambda)$ and, for instance, $(A \wedge B), \neg(A \wedge B)$ is derivable with a proof of height one (being an axiom), but e.g. $A \wedge B, \neg A, \neg B$ cannot be established by a proof of depth one (not being an axiom). It is easy to see that weakening, inversion and contraction are in fact depth-preserving admissible if only atomic axioms of the form $p, \neg p, \Gamma$ are allowed, for $p \in V$ a propositional variable. The more general form of axioms adopted in this paper allows us to simplify many constructions as we do not have to consider a congruence rule explicitly which would allow us to prove (rather than to assume as axioms) sequents of the form $\Box A, \neg \Box A, \Gamma$.

Having dealt with the structural rules, we now address our main concern: the admissibility of the cut rule. In contrast to the absorption of structural rules, we need one additional degree of freedom in that we need to allow ourselves to apply cut to a structurally smaller formula.

Definition 14. *The* size *of a formula* $A \in \mathcal{F}(\Lambda)$ *is given inductively by* $\mathsf{size}(p) = \mathsf{size}(\bot) = 1$, $\mathsf{size}(A \wedge B) = \mathsf{size}(A \vee B) = 1 + \mathsf{size}(A) + \mathsf{size}(B)$ *and* $\mathsf{size}(\heartsuit(A_1, \dots, A_n)) = 1 + \mathsf{size}(A_1) + \dots + \mathsf{size}(A_n)$.

A ruleset R absorbs cut, *if for all rules* $(r_1)\frac{\Gamma_1, \dots, \Gamma_n}{A, \Gamma_0}$, $(r_2)\frac{\Delta_1, \dots, \Delta_k}{\neg A, \Delta_0} \in \mathsf{R}$

$$\mathsf{GR} + \mathsf{Cut}(A, r_1, r_2) \vdash \Gamma_0, \Delta_0$$

where $\mathsf{Cut}(A, r_1, r_2)$ *consists of all instances of the rule schemas*

$$\frac{\Gamma, C \quad \Delta, \neg C}{\Gamma, \Delta} \qquad \frac{\Gamma}{\Gamma, A} \qquad \frac{\Gamma, A, A}{\Gamma, A} \qquad \frac{\Gamma, \neg\neg A}{\Gamma, A} \qquad \frac{\Gamma, \neg(A_1 \wedge A_2)}{\Gamma, \neg A_1, \neg A_2} \qquad \frac{\Gamma, A_1 \wedge A_2}{\Gamma, A_i}$$

where $\mathsf{size}(C) < \mathsf{size}(A)$ *in the leftmost rule and* $i = 1, 2$ *in the rightmost schema, together with the axioms* $\Gamma_1, \dots, \Gamma_n, \Delta_1, \dots, \Delta_k$ *and all sequents of the form* Γ, Δ *where* $\Gamma, \Delta \in \mathsf{S}(\Lambda)$ *and, for some* $B \in \mathcal{F}(\Lambda)$,

- Γ, B *and* $\Delta, \neg B \in \{\Gamma_1, \dots, \Gamma_n, \Delta_1, \dots, \Delta_k\}$, *or*
- $\Gamma, B = \Gamma_0, A$ *and* $\Delta, \neg B \in \{\Delta_1, \dots, \Delta_k\}$, *or*
- $\Gamma, B = \Delta_0, \neg A$ *and* $\Delta, \neg B \in \{\Gamma_1, \dots, \Gamma_n\}$.

A rule set that absorbs structural rules and the cut rule is called absorbing.

The intuition behind the above definition is similar to that of weakening, but we have two additional degrees of freedom: we can not only apply the cut rule to rule premises, but we can moreover freely use cut on structurally smaller formulas as well as structural rules. This allows us to use the standard argument, a double induction on the structure of the cut formula and the size (or height) of the proof tree, to establish cut elimination. This is carried out in the proof of the next theorem.

Theorem 15. *Suppose* R *is absorbing. Then the cut rule*

$$\frac{\Gamma, A \qquad \Delta, \neg A}{\Gamma, \Delta}$$

is admissible in GR.

We illustrate the preceding theorem by using it to derive the well-known fact that cut-elimination holds for the modal logics $\mathsf{K}, \mathsf{K4}$ and T and use it to derive cut-elimination for various conditional logics in Section 6.

Example 16. The rule sets $\mathsf{K}, \mathsf{K4}$ and T are absorbing. We have already seen that they absorb weakening, contraction and inversion in Example 11 so everything that remains to be seen is that they also absorb cut. For (K), we need to apply cut to a formula of smaller size. For the two instances

$$(r_1)\frac{\neg A_1, \dots, \neg A_n, A_0}{\neg\Box A_1, \dots, \neg\Box A_n, \Box A_0, \Gamma} \qquad (r_2)\frac{\neg B_1, \dots, \neg B_k, B_0}{\neg\Box B_1, \dots, \neg\Box B_k, \Box B_0, \Delta}$$

we need to consider, up to symmetry, the cases $A_i = B_0$, $\Box A_i \in \Delta$ and $\neg\Box A_0 \in \Delta$, for $i = 1, \dots, n$. Here, we only treat the first case for $i = 1$ where we have to show that $\neg\Box A_2, \dots, \neg\Box A_n, \Box A_0, \neg\Box B_1, \dots, \neg\Box B_k, \Gamma, \Delta$ is derivable from $\mathsf{GR} + \mathsf{Cut}(\Box A_1, r_1, r_2)$, which follows as the latter system allows us to apply cut on $A_1 = B_0$. The case $\Box A_i \in \Delta$ and $\neg\Box A_0 \in \Delta$ are straight forward.

The argument to show that $(\mathsf{K4})$ is absorbing is similar, and uses an additional (admissible) instance of cut on a formula of smaller size and contraction. For (T) we only consider instances of cut between two conclusions of

$$(r_1)\frac{\neg A, \neg\Box A, \Gamma}{\neg\Box A, \Gamma} \qquad (r_2)\frac{\neg B, \neg\Box B, \Delta}{\neg\Box B, \Delta}$$

of the T-rule. We only demonstrate the case $\Box A \in \Delta$. In this case, $\Delta = \Delta', \Box A$ and we have to show that $\neg\Box B, \Gamma, \Delta'$ can be derived in $\mathsf{Cut}(\Box A, r_1, r_2)$. The latter system allows us to cut $\neg\Box A$ between the conclusion of (T) on the left and the premise of the right hand rule, i.e., we have that $\mathsf{Cut}(\Box A, r_1, r_2) \vdash \neg B, \neg\Box B, \Gamma, \Delta')$ and an application of (T) now gives derivability of $\neg\Box B, \Gamma, \Delta'$.

5 Equivalence of Hilbert and Gentzen Systems

We now investigate the relationship between provability in a Hilbert-system and provability in the associated Gentzen system. We note the following standard lemmas that we will use later on.

Lemma 17. *Suppose* $A \in \mathcal{F}(\Lambda)$ *is a propositional tautology. Then* $\mathsf{GR} \vdash A$*. If moreover* R *is closed under substitution, then* $\mathsf{GR} \vdash \Gamma\sigma$ *whenever* $\mathsf{GR} \vdash \Gamma$ *for all* $\Gamma \in \mathsf{S}(\Lambda)$*.*

Remark 18. Being able to prove the previous lemma is the main reason for formulating axioms as $A, \neg A, \Gamma$ where $A \in \mathcal{F}(\Lambda)$ rather than $p, \neg p, \Gamma$. Both formulations are equivalent if the modal congruence rule

$$\frac{A_1 \leftrightarrow A_1' \qquad \dots \qquad A_n \leftrightarrow A_n'}{\heartsuit(A_1, \dots, A_n) \to \heartsuit(A_1', \dots, A_n')}$$

is admissible. However, Lemma 17 can be proved without the assumption that congruence is admissible using axioms of the form $A, \neg A, \Gamma$.

Theorem 19. *Suppose* R *is absorbing and substitution closed. Then* $\mathsf{GR} \vdash \Gamma \iff \mathsf{HR} \vdash \bigvee \Gamma$ *for all* $\Gamma \in \mathsf{S}(\Lambda)$*.*

Proof (Sketch). We only need to show the direction from right to left. Inductively assume that $\mathsf{HR} \vdash \bigvee \Gamma$ for $\Gamma \in \mathsf{S}(\Lambda)$. By Proposition 9 we have that there are rules Θ_i/Γ_i and substitutions σ_i, $i = 1, \dots, n$ such that

- $\mathsf{HR} \vdash \Delta\sigma_i$ whenever $\Delta \in \Theta_i$ $(i = 1, \dots, n)$
- $\{\bigvee \Gamma_1\sigma_1, \dots, \bigvee \Gamma_n\sigma_n\} \vdash_{\mathsf{PL}} \bigvee \Gamma$.

By induction hypothesis, $\mathsf{GR} \vdash \Delta\sigma_i$ for all $i = 1, \dots, n$ and $\Delta \in \Theta_i$. By Lemma 17 we have

$$\mathsf{GR} \vdash \bigvee \Gamma_1\sigma_1 \wedge \dots \wedge \bigvee \Gamma_n\sigma_n \to \bigvee \Gamma.$$

The claim follows by applying cut, contraction and inversion.

The construction of an absorbing rule set from a given set of axioms and rules essentially boils down to adding the missing instances of cut, weakening, contraction and inversion to a given rule set. The soundness of this process is witnessed by the following two simple lemmas, which we use in this section to derive an absorbing rule set for K and to establish cut-elimination for a large range of conditional logics in the next section.

Lemma 20. *Suppose* $\Gamma_1, \dots, \Gamma_n/\neg A, \Gamma_0$ *and* $\Delta_1, \dots, \Delta_k/A, \Delta_0 \in \mathsf{R}$*. Then the rule* $\Gamma_1, \dots, \Gamma_n, \Delta_1, \dots, \Delta_k/\Gamma_0, \Delta_0$ *is admissible in* HR*.*

The same applies to instances of the structural rules of weakening, contraction and inversion. As we are extending the rule set while leaving the provability predicate in the Hilbert calculus unchanged, the following formulation is handy for our purposes – in particular it implies the fact that we can freely use structural rules both in the premise and conclusion.

Lemma 21. *Suppose that* $\Gamma_1, \ldots, \Gamma_n/\Gamma_0 \in \mathsf{R}$. *If* $\Delta_0, \ldots, \Delta_1 \in \mathsf{S}(\Lambda)$ *and both*

$$\{\bigvee \Delta_1, \ldots, \bigvee \Delta_k\} \vdash_{\mathsf{PL}} \bigvee \Gamma_i (1 \le i \le n) \text{ and } \bigvee \Gamma_0 \vdash_{\mathsf{PL}} \bigvee \Delta_0$$

then the rule $\Gamma_1, \ldots, \Gamma_k/\Gamma_0$ *is admissible in* HR.

This gives us a recipe for constructing rule sets that absorb contraction and cut: simply add more rules according to the lemmas above. This will not change the notion of provability in the Hilbert system, but when this process terminates, the ensuing rule set will be absorbing and gives rise to a cut free sequent calculus.

Example 22 (Modal Logic K). In a Hilbert-style calculus, the axiomatisation of K is usually described in terms of the distribution axiom (which we view as a rule with empty premise) and the necessitation rule:

$$(\mathsf{D}) \quad \Box(A \to B) \to \Box A \to \Box B \qquad (\mathsf{N})\frac{A}{\Box A}$$

We first apply Lemma 20 to break the propositional connectives in the distribution axiom. We have that the axiom $\neg\Box(A \to B), \neg\Box A, \Box B$ is admissible by Lemma 21, and applying Lemma 20 to this axiom and the instance $A \to B/\Box(A \to B)$ of the necessitation rule gives admissibility of the rule set consisting of all instances of

$$\frac{\neg A, B}{\neg\Box A, \Box B}$$

with the help of (admissible) propositional reasoning in the premise. The same procedure, applied to the instances

$$\frac{\neg A, B \to C}{\neg\Box A, \Box(B \to C)} \qquad \neg\Box(B \to C), \neg\Box B, \Box C$$

gives admissibility of

$$\frac{\neg A, \neg B, C}{\neg\Box A, \neg\Box B, \Box C}.$$

Continuing in this way and absorbing weakening, we obtain admissibility of

$$\frac{\neg A_1, \ldots, \neg A_n, A_0}{\neg\Box A_1, \ldots, \neg\Box A_n, \Box A_0, \Gamma}$$

where $\Gamma \in \mathsf{S}(\Lambda)$ is an arbitrary context. We have shown previously that this rule set is absorbing, and it is easy to see that it is equivalent to the rule set consisting of all instances of N and D.

6 Applications: Sequent Calculi for Conditional Logics

After having seen how the construction of absorbing rule sets gives rise to standard cut-elimination results for a number of well-studied normal modal logics, in this section we construct a cut-free sequent calculus for a number of conditional logics.

Conditional logics [2] are extensions of propositional logic by a non-monotonic conditional $A \Rightarrow B$, read as "B holds under the condition that A". The conditional implication is non-monotonic in general, that is the validity of $A \Rightarrow B$ does *not* imply that also $(A \wedge C) \Rightarrow B$ is a valid statement.

Axiomatically, the first argument A of the conditional operation $A \Rightarrow B$ behaves like the $\Box$ in neighbourhood frames and only supports replacement of equivalents, whereas the second argument B obeys the rules of K. Standard conditional logic CK is axiomatised by the following two rules

$$\frac{A \leftrightarrow A'}{A \Rightarrow B \leftrightarrow A' \Rightarrow B} \qquad \frac{\bigwedge_{1 \le i \le n} B_i \to B_0}{\bigwedge_{1 \le i \le n} A \Rightarrow B_i \to (A \to B_0)}$$

in [2]. We write CK for the rule set that comprises both of the above rules. Other axioms of interest include

$$(\mathsf{ID})\, A \Rightarrow A \qquad (\mathsf{MP})\, (A \Rightarrow B) \to A \to B \qquad (\mathsf{CEM})\, (A \to B) \vee (A \Rightarrow \neg B)$$

The first axiom embodies a form of identity in the sense that A holds under condition A and (MP) is a conditional form of modus ponens. The axiom (CEM) is the conditional excluded middle. We denote combination of rule sets by juxtaposition so that CKID comprises all instances of CK and ID.

6.1 Cut Elimination for Extensions of CK without CEM

We first treat extensions of the basic conditional logic CK with axioms ID and MP, but not including CEM and discuss CEM later, as the effect of adding CEM leads to a more general form of the CK rule. We start by introducing some notation that provides a shorthand for expressing the bi-implications in the premise of CK.

Notation 23. If $A_0, \dots, A_n \in \mathcal{F}(\Lambda)$ are conditional formulas, we write $A_0 = \dots = A_n$ for the sequence of sequents consisting of $\neg A_0, A_i$ and $\neg A_i, A_0$ for all $1 \le i \le n$.

If we absorb cuts using Lemmas 20 and 21 we see that all instances of

$$(\mathsf{CK}_g)\frac{A_0 = \dots = A_n \quad \neg B_1, \dots, \neg B_n, B_0}{\neg(A_1 \Rightarrow B_1), \dots, \neg(A_n \Rightarrow B_n), (A_0 \Rightarrow B_0), \Gamma}$$

are admissible in HCK. It is easy to see that the rule set CK_g is actually absorbing:

Theorem 24. *The rule set CK_g is absorbing and equivalent to CK. As a consequence, GCK_g has cut-elimination and $\mathsf{GCK}_g \vdash A$ iff $\mathsf{HCK} \vdash A$ whenever $A \in \mathcal{F}(\Lambda)$.*

Proof. Using Lemmas 20 and Lemma 21 it is immediate that the rule set CK_g is admissible in HCK. The argument that show that CK_g is absorbing is analogous to that for the modal logic K (Example 16), and the result follows from Theorem 19.

The logic CKID arises form CK by adding the identity axiom $A \Rightarrow A$ to the rule set CK_H that axiomatises standard conditional logic. Applying Lemma 20 to the two rule instances on the left

$$\frac{\neg A, B \quad \neg B, A}{\neg(A \Rightarrow A), (A \Rightarrow B)} \qquad A \Rightarrow A \qquad (\mathsf{ID}_g)\frac{A = B}{A \Rightarrow B, \Gamma}$$

gives rise to the rule schema (ID) on the right where we have used Lemma 21 to absorb weakening. If we denote the rule set consisting of all instances of CK_g and ID_g by CKID_g, we obtain:

Proposition 25. *The rule set* CKID_g *is absorbing and equivalent to* CKID.

Proof. It is easy to see that CKID_g absorbs the structural rules, and that CKID is equivalent to CKID_g. Cuts between conclusions of (ID_g) are readily seen to be absorbed, and absorption of cuts between an instance of CK_g and an instance of ID follows by construction.

The logic CKMP arises by augmenting the logic CK with the additional axiom $(A \Rightarrow B) \to (A \to B)$. The effect of adding ($\mathsf{MP}$) is similar to that of enriching the modal logic K with the (T)-axiom. Adding the missing cuts to CK augmented with (MP) and absorbing the structural rules leads to the rule schema

$$(\mathsf{MP}_g)\frac{A, \neg(A \Rightarrow B), \Gamma \quad \neg B, \neg(A \Rightarrow B), \Gamma}{\neg(A \Rightarrow B), \Gamma}$$

and we denote the rule set consisting of all instances of CK_g and MP_g by CKMP_g. Our cut elimination theorem then takes the following form:

Proposition 26. *The rule set* CKMP_g *is absorbing and equivalent to* CKMP.

Proof. Again, it is easy to see that CKMP_g is admissible in HCKMP and the converse follows by construction. All we have to show is that CKMP_g is absorbing, where the absorption of structural rules is easy and left to the reader. For the absorption of cut, the argument is similar to cut elimination in the modal logic T, and can be found in the appendix.

6.2 Cut Elimination for Extensions of CKCEM

To construct an absorbing rule set for conditional logic plus the axiom

$$(\mathsf{CEM})(A \Rightarrow B) \vee (A \Rightarrow \neg B)$$

we start from the admissible rule set for CK and close under cuts that arise with (CEM). Repeated applications of Lemma 20 and Lemma 21 lead to the rule set

$$(\mathsf{CEM}_g)\frac{A_0 = \cdots = A_n \quad B_0, \ldots, B_j, \neg B_{j+1}, \neg B_n}{(A_0 \Rightarrow B_0), \ldots, (A_j \Rightarrow B_j), \neg(A_{j+1} \Rightarrow B_{j+1}), \ldots, \neg(A_n \Rightarrow B_n), \Gamma}$$

for $1 \leq j \leq n$.

Proposition 27. *The rule set* CKCEM_g *is absorbing and equivalent to* CKCEM.

As a consequence, cut elimination holds in CKCEM_g. We can apply essentially the same argument to an extension of CK with both conditional modus ponens and conditional excluded middle, but have to take care of the cuts arising between MP_g and CEM_g, which leads to the new rule

$$(\mathsf{MPEM}_g)\frac{A, (A \Rightarrow B), \Gamma \qquad B, (A \Rightarrow B), \Gamma}{(A \Rightarrow B), \Gamma}.$$

If we denote the extension of CKCEM_g with MP_g and MPEM_g by $\mathsf{CKCEMMP}_g$, we obtain:

Proposition 28. *The rule set* $\mathsf{CKCEMMP}_g$ *is absorbing and equivalent to* $\mathsf{CKCEMMP}$.

We note that the latter theorem was left as an open problem for the sequent system presented in [8]. In summary, we obtain the following results about extensions of the conditional logic CK.

Theorem 29. *Suppose that* L *is one of* CK, CKID, CKMP, CKCEM *or* $\mathsf{CKCEMMP}$. *Then* $\mathsf{GL}_g \vdash A$ *whenever* $\mathsf{HL} \vdash A$ *for all* $A \in \mathcal{F}(\Lambda)$. *Moreover, cut elimination holds in* GL.

The theorem follows, in each of the cases, from Theorem 15 and Theorem 19 together with the fact that the rule set Ł and $Ł_g$ are equivalent and the latter is absorbing.

7 Complexity of Proof Search

It is comparatively straightforward to extract complexity bounds for provability of the logics considered above by analysing the complexity of proof search under suitable strategies in the cut-free sequent systems obtained. Clearly, in those cases where all modal rules peel off exactly one layer of modal operators, the depth of proofs is polynomial in the nesting depth of modal operators in the target formula, and therefore, proof search is in PSPACE under mild assumptions on the branching width of proofs [12, 9]. Besides reproving Ladner's classical result for K [7], we thus have

Theorem 30. *Provability in* CK *and* CKID *is in PSPACE.*

This reproves known complexity bounds originally shown in [8] (alternative short proofs using coalgebraic semantics are given in [11]). For CKCEM, the bound can be improved using dynamic programming in the same style as in [13]:

Theorem 31. *Provability in* CKCEM *is in coNP.*

More interesting are those cases where some of the modal operators from the conclusion remain in the premise, such as T, $\mathsf{K4}$, CKMP, and CKCEM (where the difference between non-iterative logics, i.e. ones whose Hilbert-axiomatisation does not use nested modalities, such as T, CKMP, and $\mathsf{CKMPCEM}$, and iterative logics such as $\mathsf{K4}$ is surprisingly hard to spot in the sequent presentation). For $\mathsf{K4}$, the standard approach is to consider proofs of minimal depth, which therefore never attempt to prove a sequent repeatedly, and analyse the maximal depth that a branch of a proof can have without repeating a sequent. For T, a different strategy is used, where the (T) rule is limited to be applied at most once to every formula of the form $\neg\Box A$ in between two applications of (K) [6]. A similar strategy works for the conditional logics CKMP and $\mathsf{CKMPCEM}$, which we explain in some additional detail for CKMP.

We let CKMP_g^0 and CKMP_g^1 denote restricted sequent systems where in CKMP_g^0, a formula $\neg(A \Rightarrow B)$ is marked on a branch as soon as the rule (MP_g) has been applied to it (backwards) and unmarked only at the next application of rule (CK_g). Rule (MP_g) applies only to unmarked formulas. In CKMP_g^1, we instead impose a similar restriction where rule (MP_g) applies to a sequent $\neg(A \Rightarrow B), \Gamma$ only in case Γ does not contain a propositional descendant of either A or $\neg B$. Here, a sequent Δ is called a *propositional descendant* of a formula A if it can be generated from A by applying propositional sequent rules backwards (e.g. the propositional descendants of $(\neg(A \wedge B) \wedge C)$ are $\neg(A \wedge B)$; C; and $\neg A, \neg B$). It is easy to check that CKMP_g^1-proofs can be converted into CKMP_g^0-proofs, i.e. CKMP_g^1 is the most restrictive system. One shows that CKMP_g^1 admits contraction and inversion by verifying that the corresponding proof transformations in CKMP_g preserve CKMP_g^1-proofs. It is then clear that every application of the rule (MP_g) that violates the CKMP_g^1-restriction can be replaced by inversion and contraction, so that CKMP_g^1, and hence also CKMP_g^0, proves the same formulas as CKMP_g. Proofs in CKMP_g^0 are easily seen to have at most polynomial depth. Essentially the same reasoning applies to $\mathsf{CKMPCEM}$. Therefore, we have

Theorem 32. *Provability in* CKMP *is in PSPACE; provability in* $\mathsf{CKMPCEM}$ *is in coNP.*

We note that the complexity of $\mathsf{CKMPCEM}$ was explicitly left open in [8].

8 Conclusions

We have established a generic method of cut elimination in modal sequent system based on absorption of cut and structural rules by sets of modal rules. We have applied this method in particular to various conditional logics, thus obtaining cut-free unlabelled sequent calculi that complement recently introduced labelled calculi [8]. In at least one case, the conditional logic $\mathsf{CKMPCEM}$ with modus ponens and conditional excluded middle, our calculus seems to be the first cut-free calculus in the literature, as cut elimination for the corresponding calculus in [8] was explicitly left open. We have applied these calculi to obtain complexity bounds on proof search in conditional logics; in particular we have reproved known upper complexity bounds for CK, CKID, CKMP [8] and improved the bound for CKCEM from PSPACE to coNP using dynamic programming techniques following [13]. Moreover, we have obtained an upper bound coNP for CKMPCEM, for which no bound has previously been published. We conjecture that our general method can also be applied to other base logics, e.g. intuitionistic propositional logic or first-order logic, which is subject to further investigations.

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Appendix: Omitted Proofs

Proof of Theorem 15

Proof. We use Gentzen's classical method and proceed by a double induction on the size of the cut formula and the height of the proof tree. That is, we prove the statement

$$\forall A \in \mathcal{F}(\Lambda) \forall n \in \omega (n = n_1 + n_2 \& \vdash_{n_1} \Gamma, A, \vdash_{n_2} \Delta, \neg A \implies \vdash \Gamma, \Delta)$$

by induction on $\mathsf{size}(A)$ where, in the inductive step, we use a side induction the size of proof trees, as indicated by the subscript of the entailment sign. Formally, the relation $\vdash_n$ is defined inductively by $\vdash_1 \Gamma, A, \neg A$ and

$$\frac{\vdash_n A}{\vdash_{n+1} \neg\neg A} \qquad \frac{\vdash_n \Gamma, \neg A, \neg B}{\vdash_{n+1} \Gamma, \neg(A \wedge B)} \qquad \frac{\vdash_n \Gamma, A \quad \vdash_k \Gamma, B}{\vdash_{n+k+1} \Gamma, A \wedge B} \qquad \frac{\vdash_{n_1} F_1 \ldots \vdash_{n_k} F_k}{\vdash_{n_1+\cdots+n_k+1} F_0}$$

where, in the last rule, $\frac{F_1 \ldots F_k}{F_0} \in \mathsf{R}$. We may inductively assume that the statement holds for all cut formulas $C < A$ and to prove the statement for A we have to consider the following cases:

1. cuts that arise between two rule conclusions
2. cuts that arise between a rule conclusion and the conclusion of a propositional rule or axiom
3. cuts that arise between two propositional rules.

We start with item (1), which follows directly from the fact that R absorbs cut. In more detail, suppose that $\frac{F_1, \ldots, F_k}{F_0, A}$ and $\frac{G_1 \ldots G_k}{G_0, \neg A} \in \mathsf{R}$ and $\vdash_{n_i} F_i$ $(i = 1, \ldots, k)$ and $\vdash_{m_j} G_j$ for $j = 1, \ldots, l$. As R absorbs cut, we have that F_0, G_0 is derivable using cuts on formulas $< A$ from the additional assumptions Γ, Δ provided that for some $D \in \mathcal{F}(\Lambda)$ we have that both Γ, D and $\Delta, \neg D$ are among the $F_1, \ldots, F_k, G_1, \ldots, G_l$. In case $\Gamma, D = \Delta, \neg D$ we have that $\Gamma \subseteq \Gamma, \Delta$ and $\vdash \Gamma, \Delta$ as weakening is admissible in GR. Assuming that $\vdash_x \Gamma, D$ and $\vdash_y \Delta, \neg D$ for $\Gamma, D \neq \Delta, \neg D$ we have that $x + y < 2 + \sum_i n_i + \sum_j m_j$ and hence $\vdash \Gamma, \Delta$ by (inner) induction hypothesis. The fact that – in the deduction of F_0, G_0 – we may also have to use cuts on formulas $< A$ is discharged by the outer induction hypothesis and possible uses of weakening, contraction and inversion are admissible by Proposition 12.

As regards item (2) we only discuss a subset of the cases that showcase the need for contraction, weakening and inversion to be admissible. For the whole discussion, suppose that $\frac{F_1 \ldots F_k}{F_0} \in \mathsf{R}$ and $\vdash_{n_i} F_i$ for $i = 1, \ldots, k$.

- Suppose that $F_0 = F_0', A$ and $G_0, \neg A$ is an axiom. In case $A \in G_0$ we have that $F_0 = F_0', A \subseteq F_0', G_0$ and $\vdash F_0', G_0$ follows from $\vdash F_0', A$ as GR admits weakening. In case $\neg A \notin G_0$ we have that G_0 is an axiom, and hence so is G_0, F_0'.

- Suppose that $F_0 = F_0', A$ and $\neg A, G_0$ has been derived using $(\neg \bigwedge)$. We have to discuss two cases, depending on whether or not $\neg A$ is principal in the application of $(\neg\wedge)$.
  Case $A = A' \wedge B'$ *and* $\vdash_m \neg A', \neg B', G_0$ *so that* $\vdash_{m+1} \neg A, G_0$. As R absorbs structural rules, we have that $\mathsf{GR} \vdash F_0', A$ and $\mathsf{GR} \vdash F_0', B$. As cuts on A' and B' can be eliminated by induction hypothesis, we have $\mathsf{GR} \vdash F_0', F_0', G_0$ and therefore $\mathsf{GR} \vdash F_0', G_0$ as GR admits contraction.
  Case $\vdash_m \neg C, \neg D, \neg A, G_0$ *so that* $\vdash_{m+1} \neg(C \wedge D), \neg A, G_0$. As $m + 1 + \sum_{i=1}^{k} n_i < m+1+1+\sum_{i=1}^{k} n_i$ we may apply the inner induction hypothesis to conclude $\vdash \neg C, \neg D, G_0, F_0'$ and applying $(\neg\wedge)$ gives $\vdash F_0', \neg(C \wedge D), G_0$.

All the other cases follow exactly the same pattern. We now focus on item (3), that is, we show how cuts between the conclusions of propositional rules and axioms can be eliminated. This is mostly standard and again we only discuss a subset of the cases. Suppose that $\vdash_n F_0, A$ and $\vdash_m G_0, \neg A$.

- If both F_0, A and $G_0, \neg A$ are axioms, then so is F_0, G_0.
- Suppose that F_0, A has been derived using $(\wedge)$ and $G_0, \neg A$ has been derived using $(\neg\wedge)$. We distinguish four cases depending on whether or not A is principal in the application of $\wedge$ or $(\neg\wedge)$.
  Case $A = A' \wedge B'$ *and* $\vdash_{n_0} F_0, A$, $\vdash_{n_1} F_0, B'$ *so that* $n = n_0 + n_1 + 1$ *and* $\vdash_n A \wedge B, F_0$. If A is principal in the application of $(\neg\wedge)$, we have that $\vdash_{m-1} G_0, \neg A', \neg B'$. By (outer) induction hypothesis, cuts on A' and B' can be eliminated so that we have $\vdash F_0, F_0, G_0$ and it follows from closure under contraction that $\vdash F_0, G_0$.
  If A is not principal in the application of $(\neg\wedge)$ we have that $\vdash_{m-1} \neg C, \neg D, \neg(A' \wedge B'), G_0'$ so that $G_0 = \neg(C \wedge D), G_0'$ and $\vdash_m \neg(A \wedge B), G_0'$. As $\vdash_n F_0, A$ and $\vdash_{m-1} \neg C, \neg D, \neg A, G_0'$ and $n + (m-1) < n + m$ we can apply the inner induction hypothesis to eliminate the cut on A so that $\vdash F_0, \neg C, \neg D, G_0'$ and applying $(\neg\wedge)$ gives $\vdash F_0, \neg(C \wedge D), G_0' = F_0, G_0$ as required.
  The two cases where A is not principal in the application of $(\neg\wedge)$ follow exactly the same pattern.

The remaining cases of cuts between propositional rules and axioms are entirely analogous, and therefore omitted.

Proof of Proposition 26

Proof. It is clear that both (CK_g) and (MP_g) absorb the structural rules. For cut, we first consider cuts between two instances of (MP_g), say

$$(r_1)\frac{A, \neg(A \Rightarrow B), \Gamma \quad \neg B, \neg(A \Rightarrow B), \Gamma}{\neg(A \Rightarrow B), \Gamma} \qquad (r_2)\frac{C, \neg(C \Rightarrow D), \Delta \quad \neg D, \neg(C \Rightarrow D), \Delta}{\neg(C \Rightarrow D), \Delta}$$

where the cut happens on $F \in \mathcal{F}(\Lambda)$. We distinguish several cases:

Case $F = (A \Rightarrow B)$ *and* $F \in \Delta$. Then $\Delta = (A \Rightarrow B), \Delta'$ for some $\Delta' \in \mathsf{S}(\Lambda)$. To eliminate the cut on C, we note that the following two derivations

$$\dfrac{\dfrac{A, \neg(A \Rightarrow B), \Gamma \quad \neg B, \neg(A \Rightarrow B), \Gamma}{\neg(A \Rightarrow B), \Gamma}\,(\mathsf{MP}) \qquad C, \neg(C \Rightarrow D), (A \Rightarrow B), \Delta'}{C, \neg(C \Rightarrow D), \Gamma, \Delta'}\,(\text{cut } (F))$$

and

$$\dfrac{\dfrac{A, \neg(A \Rightarrow B), \Gamma \quad \neg B, \neg(A \Rightarrow B), \Gamma}{\neg(A \Rightarrow B), \Gamma}\,(\mathsf{MP})) \qquad \neg D, \neg(C \Rightarrow D), (A \Rightarrow B), \Delta'}{\neg D, \neg(C \Rightarrow D), \Gamma, \Delta'}\,(\text{cut } (F))$$

witness that we can use both $C, \neg(C \Rightarrow D), \Gamma, \Delta'$ and $\neg D, \neg(C \Rightarrow D), \Gamma, \Delta'$ as axioms in $\mathsf{CKMP}_g + \mathsf{Cut}(F, r_1, r_2)$ as the cuts occur between the premises of (r_2) and conclusions of (r_1). Applying (MP_g) to these axioms, we obtain that $\mathsf{CKMP}_g + \mathsf{Cut}(F, r_1, r_2) \vdash \neg(C \Rightarrow D), \Gamma, \Delta'$.

Case $F = (C \Rightarrow D)$ *and* $F \in \Gamma$. This is symmetric to the case above.

Case $F \in \Gamma$ *and* $\neg F \in \Delta$. Then $\Gamma = \Gamma', F$ and $\Delta = \Delta', \neg F$. We have to show that

$$\neg(A \Rightarrow B), \neg(C \Rightarrow D), \Gamma', \Delta'$$

is derivable in $\mathsf{Cut}(F, r_1, r_2)$. We note that the deduction

$$\dfrac{A, \neg(A \Rightarrow B), F, \Gamma' \qquad C, \neg(C \Rightarrow D), \neg F, \Delta'}{A, \neg(A \Rightarrow B), C, \neg(C \Rightarrow D), \Gamma', \Delta'}$$

witnesses that we may use $A, \neg(A \Rightarrow B), C, \neg(C \Rightarrow D), \Gamma', \Delta'$ as an axiom in the system $\mathsf{CKMP}_g + \mathsf{Cut}(F, r_1, r_2)$ as the cut on F has occurred between premises of r_1 and r_2. The same deduction, with C replaced by $\neg D$ throughout, witnesses that this is also the case for $A, \neg(A \Rightarrow B), \neg D, \neg(C \Rightarrow D), \Gamma', \Delta'$. An application of (MP_g) now yields $\mathsf{CKMP}_g + \mathsf{Cut}(F, r_1, r_2) \vdash \neg(C \Rightarrow D), A, \neg(A \Rightarrow B), \Gamma', \Delta'$.

By the symmetric argument (replace A by $\neg B$) we obtain that also $\mathsf{CKMP}_g + \mathsf{Cut}(F, r_1, r_2) \vdash \neg(C \Rightarrow D), \neg B, \neg(A \Rightarrow B), \Gamma', \Delta'$ and an application of (MP_g) now yields $\mathsf{CKMP}_g + \mathsf{Cut}(F, r_1, r_2) \vdash \neg(C \Rightarrow D), \neg(A \Rightarrow B), \Gamma', \Delta'$ as required.

What is left is to consider cuts, say on $F \in \mathcal{F}(\Lambda)$, between the conclusions of the rules

$$(r_1)\,\dfrac{A_0 = \cdots = A_n \quad \neg B_1, \ldots, \neg B_n, B_0}{\neg(A_1 \Rightarrow B_1), \ldots, \neg(A_n \Rightarrow B_n), (A_0 \Rightarrow B_0), \Gamma}$$

$$(r_2)\,\dfrac{C, \neg(C \Rightarrow D), \Delta \quad \neg D, \neg(C \Rightarrow D), \Delta}{\neg(C \Rightarrow D), \Delta}$$

As before, we need to discuss several cases.

Case $F \in \Gamma$ *or* $\neg F \in \Gamma$. Trivial, as the conclusion of the cut can be derived using a different weakening context Γ.

Case $F = (A_i \Rightarrow B_i)$ *for* $1 \leq i \leq n$. We assume without loss of generality that $i = 1$ and have that $F = (A_1 \Rightarrow B_1) \in \Delta$ so that $\Delta = \Delta', F$. To replace the cut on F, we consider the deduction

$$\dfrac{\dfrac{A_0 = \cdots = A_n \quad \neg B_1, \ldots, \neg B_n, B_0}{\neg(A_1 \Rightarrow B_1), \ldots, \neg(A_n \Rightarrow B_n), (A_0 \Rightarrow B_0, \Gamma)}\ (\text{CK}) \qquad C, \neg(C \Rightarrow D), (A_1 \Rightarrow B_1), \Delta'}{\neg(A_2 \Rightarrow B_2), \ldots, \neg(A_n \Rightarrow B_n), (A_0 \Rightarrow B_0), C, \neg(C \Rightarrow D), \Gamma, \Delta'}\ (\text{cut } (F))$$

which witnesses that we may use

$$\Sigma_1 = \neg(A_2 \Rightarrow B_2), \ldots, \neg(A_n \Rightarrow B_n), (A_0 \Rightarrow B_0), C, \neg(C \Rightarrow D), \Gamma, \Delta'$$

as an axiom in $\mathsf{GCKMP}_g + \mathsf{Cut}(F, r_1, r_2)$. The above deduction, with C replaced by $\neg D$ throughout, witnesses that the same is true for

$$\Sigma_2 = \neg(A_2 \Rightarrow B_2), \ldots, \neg(A_n \Rightarrow B_n), (A_0 \Rightarrow B_0), \neg D, \neg(C \Rightarrow D), \Gamma, \Delta'$$

and applying (MP_g) with premises Σ_1 and Σ_2 yields $\mathsf{GCKMP}_g + \mathsf{Cut}(F, r_1, r_2) \vdash \neg(A_2 \Rightarrow B_2), \ldots, \neg(A_n \Rightarrow B_n), (A_0 \Rightarrow B_0), \neg(C \Rightarrow D), \Gamma, \Delta'$ as required.

Case $F = (A_0 \Rightarrow B_0) = (C \Rightarrow D)$. We have to give a derivation of $\neg(A_1 \Rightarrow B_1), \ldots, \neg(A_n \Rightarrow B_n), \Gamma, \Delta$ in $\mathsf{Cut}(F, r_1, r_2)$. The deduction

$$\dfrac{\dfrac{A_0 = \cdots = A_n \qquad \neg B_1, \ldots, \neg B_n, B_0}{\neg(A_1 \Rightarrow B_1), \ldots, \neg(A_n \Rightarrow B_n), (A_0 \Rightarrow B_0), \Gamma}\ (\text{CK}) \qquad \neg B_0, \neg(A_0 \Rightarrow B_0), \Delta}{\neg(A_1 \Rightarrow B_1), \ldots, \neg(A_n \Rightarrow B_n), \neg B_0, \Gamma, \Delta}\ (\text{Cut } (F))$$

witnesses that we may use

$$\Sigma_1 = \neg(A_1 \Rightarrow B_1), \ldots, \neg(A_n \Rightarrow B_n), \neg B_0, \Gamma, \Delta$$

as an axiom in $\mathsf{GCKMP}_g + \mathsf{Cut}(F, r_1, r_2)$ as the cut on F occurs between a conclusion of (r_1) and a premise of (r_2). The same derivation, with $\neg B_0$ replaced by A_0 shows that the same is true for

$$\Sigma_2 = \neg(A_1 \Rightarrow B_1), \ldots, \neg(A_n \Rightarrow B_n), A_0, \Gamma, \Delta.$$

We therefore have the two derivations

$$\dfrac{\Sigma_1 \qquad \neg B_1, \ldots, \neg B_n, B_0}{\neg(A_1 \Rightarrow B_1), \ldots, \neg(A_n \Rightarrow B_n), \neg B_1, \ldots, \neg B_n, \Gamma, \Delta}\ (\text{Cut } (B_0))$$

and

$$\dfrac{\Sigma_2 \qquad \dfrac{\neg A_0, A_1}{\neg A_0, A_1, B_2, \ldots, B_n}\ (\text{w})}{\neg(A_1 \Rightarrow B_1), \ldots, \neg(A_n \Rightarrow B_n), A_1, B_2, \ldots, B_n, \Gamma, \Delta}\ (\text{Cut } (A_0))$$

in $\mathsf{GCKMP}_g + \mathsf{Cut}(F, r_1, r_2)$. Applying (MP_g) to the conclusions of both yields that

$$\mathsf{GCKMP}_g + \mathsf{Cut}(F, r_1, r_2) \vdash \neg(A_1, \Rightarrow B_1), \ldots, \neg(A_n \Rightarrow B_n), \neg B_2, \ldots, \neg B_n, \Gamma, \Delta.$$

Iterating the same scheme, where we use weakening on a successively smaller subset of $B_2, \ldots, B_n$ finaly yields the claim $\mathsf{GCKMP}_g + \mathsf{Cut}(F, r_1, r_2) \vdash \neg(A_1 \Rightarrow B_1), \ldots, \neg(A_n \Rightarrow B_n), \Gamma, \Delta$. Note that weakening and cuts on formulas of size $< \mathsf{size}(F)$ is admissible in $\mathsf{GCKMP}_g + \mathsf{Cut}(F, r_1, r_2)$.

Case $F = (A_0 \Rightarrow B_0)$ *and* $\neg F \in \Delta$. We have that $\Delta = \neg(A_0 \Rightarrow B_0), \Delta'$ and the deduction

$$\dfrac{\dfrac{A_0 = \cdots = A_n \qquad \neg B_1, \ldots, \neg B_n, B_0}{\neg(A_1 \Rightarrow B_1), \ldots, \neg(A_n \Rightarrow B_n), (A_0 \Rightarrow B_0), \Gamma}\,(\mathrm{CK}) \qquad C, \neg(C \Rightarrow D), \neg(A_0 \Rightarrow B_0), \Delta'}{\neg(A_1 \Rightarrow B_1), \ldots, \neg(A_n \Rightarrow B_n), C, \neg(C \Rightarrow D), \Gamma, \Delta'}\,(\mathrm{Cut}\ (F)$$

witnesses that we may use

$$\Sigma_2 = \neg(A_1 \Rightarrow B_1), \ldots, \neg(A_n \Rightarrow B_n), C, \neg(C \Rightarrow D), \Gamma, \Delta'$$

as an axiom in $\mathsf{GCKMP}_g + \mathsf{Cut}(F, r_1, r_2)$. The same derivation, with C replaced by $\neg D$, shows that the same is true for

$$\Sigma_2 = \neg(A_1 \Rightarrow B_1), \ldots, \neg(A_n \Rightarrow B_n), \neg D, \neg(C \Rightarrow D), \Gamma, \Delta'$$

and applying (MP_g) with premises Σ_1 and Σ_2 yields the claim $\mathsf{GCKMP}_g + \mathsf{Cut}(F, r_1, r_2) \vdash \neg(A_1 \Rightarrow B_1), \ldots, \neg(A_n \Rightarrow B_n), \Gamma, \Delta'$. This finishes our analysis of cuts that may arise between conclusions of the (CK_g) and the (MP_g)-rule, and hence the proof.

Proof of Proposition 27

Proof. Again, it suffices to check that the rule set CKCEM_g is absorbing, where the absorption of structural rules is clear. It therefore suffices to treat instances of cuts between conclusions of rules of CKCEM. Owing to the form of the CKCEM_g-rule, our argument is very similar to that used for CK_g. We consider the following two instances

$$(r_1)\dfrac{A_0 = \cdots = A_n \quad B_0, \ldots, B_i, \neg B_{i+1}, \ldots, \neg B_n}{(A_0 \Rightarrow B_0), \ldots, (A_i \Rightarrow B_i), \neg(A_{i+1} \Rightarrow B_{i+1}), \ldots, \neg(A_n \Rightarrow B_n), \Gamma}$$

$$(r_2)\dfrac{C_0 = \cdots = C_m \quad D_0, \ldots, D_j, \neg D_{j+1}, \ldots, \neg D_m}{(C_0 \Rightarrow D_0), \ldots, (C_j \Rightarrow D_j), \neg(C_{j+1} \Rightarrow D_{j+1}), \ldots, \neg(C_m \Rightarrow D_m), \Delta}$$

and assume that the conclusions permit an instance of cut on $F \in \mathcal{F}(\Lambda)$. As usual, we distinguish several cases, where the cases $F \in \Gamma, F \in \Delta, \neg F \in \Gamma$ and $\neg F \in \Delta$ are trivial.

Case $F = (A_k \Rightarrow B_k) = (C_l \Rightarrow D_l)$ for $k > i$ and $0 \leq l \leq j$. Without loss of generality we assume that $k = n$ and $l = 0$ and get $A_n = C_0$ and $B_n = D_0$. Denote the sequent that arises from applying cut on F to the conclusions of r_1 and r_2 by Σ_0 and notice that, using cuts on $A_n \equiv C_0$, we have that

$$\Sigma = A_0 = A_1 = \cdots = A_{n-1} = C_1 = \cdots = C_m$$

is derivable in $\mathsf{GCKCEM}_g + \mathsf{Cut}(F, r_1, r_2)$. This feeds into the derivation

$$\dfrac{\dfrac{B_0, \ldots, B_i, \neg B_{i+1}, \ldots, \neg B_n \qquad D_0, \ldots, D_j, \neg D_{j+1}, \ldots, \neg D_m}{B_0, \ldots, B_i, D_1, \ldots, D_j, \neg B_{i+1}, \ldots, \neg B_{n-1}, \neg D_{j+1}, \ldots, \neg D_m}\,(\mathrm{Cut}\ (B_n)) \qquad \Sigma}{\Sigma_0}\,(\mathsf{CKCEM})$$

which establishes that

The case $F = (A_k \Rightarrow B_k) \equiv (B_l \Rightarrow D_l)$ for $k > i$ and $1 \leq l \leq j$ is symmetric, which finishes the proof.

Proof of Proposition 28

Proof. It is clear that the rule set $\mathsf{CKCEMMP}_g$ absorbs the structural rules and it is easy to see that it is equivalent to $\mathsf{CKCEMMP}$. We have to show that it absorbs cut.

Cuts between the conclusions of two instances of MP_g have already been treated in the proof of Theorem 26, and the proof translates verbatim to cuts between instances of MPEM_g. We consider cuts between two instances

$$(r_1)\frac{A,(A\Rightarrow B),\Gamma \qquad B,(A\Rightarrow B),\Gamma}{(A\Rightarrow B),\Gamma} \qquad (r_2)\frac{C,\neg(C\Rightarrow D),\Delta \qquad \neg D,\neg(C\Rightarrow D),\Delta}{\neg(C\Rightarrow D),\Delta}$$

where the cut is performed on $F \in \mathcal{F}(\Lambda)$, say. The cases where either $F \in \Gamma$ and $\neg F$ in Δ or $F \in \Delta$ and $\neg F \in \Gamma$ are straightforward.

Case $F = (A \Rightarrow B) = (C \Rightarrow D)$. The derivation

$$\dfrac{\dfrac{A,(A\Rightarrow B),\Gamma \qquad B,(A\Rightarrow B),\Gamma}{(A\Rightarrow B),\Gamma}\,(\mathsf{MPEM}) \qquad \neg B,\neg(A\Rightarrow B),\Delta}{\neg B,\Gamma,\Delta}\,(\text{Cut }(A\Rightarrow B))$$

witnesses that we may use $\Sigma_1 = \neg B,\Gamma,\Delta$ as an axiom in $\mathsf{GCKCEMMP} + \mathsf{Cut}(F,r_1,r_2)$. Similarly, the derivation

$$\dfrac{\dfrac{A,\neg(A\Rightarrow B),\Delta \qquad \neg B,\neg(A\Rightarrow B),\Delta}{\neg(A\Rightarrow B),\Delta}\,(\mathsf{MP}) \qquad B,(A\Rightarrow B),\Gamma}{B,\Gamma,\Delta}\,(\text{Cut }(A\Rightarrow B))$$

shows that the same is true for $\Sigma_2 = B,\Gamma,\Delta$: note that in both cases, the cut was performed between an axiom and a conclusion of both rules. As $\mathsf{size}(B) < \mathsf{size}(A \Rightarrow B)$ we may now use cut on B to establish that $\mathsf{GCKCEMMP}_g + \mathsf{Cut}(F,r_1,r_2) \vdash \Gamma,\Delta$.

Case $F = (A \Rightarrow B)$ *and* $\neg F \in \Delta$. We have that $\Delta = \neg(A \Rightarrow B),\Delta'$. The derivation

$$\dfrac{\dfrac{A,(A\Rightarrow B),\Gamma \qquad B,(A\Rightarrow B),\Gamma}{(A\Rightarrow B),\Gamma}\,(\mathsf{MPEM}) \qquad C,\neg(C\Rightarrow D),\neg(A\Rightarrow B),\Delta'}{C,\neg(C\Rightarrow D),\Gamma,\Delta'}\,(\text{Cut }(F))$$

witnesses that we may use

$$\Sigma_1 = C,\neg(C\Rightarrow D),\Gamma,\Delta'$$

as an axiom in $\mathsf{GCKCEMMP} + \mathsf{Cut}(F,r_1,r_2)$. The same derivation, with C replaced by $\neg D$ shows that the same is true for

$$\Sigma_2 = \neg D,\neg(C\Rightarrow D),\Gamma,\Delta'$$

and an application of MP_g yields derivability of $\neg(C \Rightarrow D), \Gamma, \Delta'$.

Case $F = (C \Rightarrow D)$ *and* $\neg F \in \Gamma$. Analogous by interchaning the role of MP_g and MPEM_g.

This leaves to consider cuts between two instances of CKCEM_g and MP_g and between CKCEM_g and MPEM_g. We first consider the rules

$$(r_1)\frac{A, (A \Rightarrow B), \Gamma \qquad B, (A \Rightarrow B), \Gamma}{(A \Rightarrow B), \Gamma}$$

$$(r_2)\frac{A_0 = \cdots = A_n \quad B_0, \ldots, B_j, \neg B_{j+1}, \ldots, \neg B_n}{A_0 \Rightarrow B_0, \ldots, (A_j \Rightarrow B_j), \neg(A_{j+1} \Rightarrow B_{j+1}), \ldots, \neg(A_n \Rightarrow B_n), \Delta}$$

In this setting, all cases except the case $F = (A \Rightarrow B) = (A_i \Rightarrow B_i)$ with $i > j$ are entirely analogous to those considered in the proof of Theorem 26 where applications of CK_g need to be replaced by applications of CKCEM_g. For the case $F = (A \Rightarrow B) = (A_i \Rightarrow B_i)$ with $i > j$ we assume without loss of generality that $i = n$ and argue that, as in the proof of Theorem 26, that both

$$\Sigma_1 = (A_0 \Rightarrow B_0), \ldots, (A_j \Rightarrow B_j), \neg(A_{j+1} \Rightarrow B_{j+1}), \ldots, \neg(A_{n-1} \Rightarrow B_{n-1}), B_n, \Gamma, \Delta$$

and

$$\Sigma_2 = (A_0 \Rightarrow B_0), \ldots, (A_j \Rightarrow B_j), \neg(A_{j+1} \Rightarrow B_{j+1}), \ldots, \neg(A_{n-1} \Rightarrow B_{n-1}), A_n, \Gamma, \Delta$$

are axioms of $\mathsf{GCKCEMMP}_g + \mathsf{Cut}(F, r_1, r_2)$, leading to deductions ending in $\Sigma_1, B_0, \ldots, B_j, \neg B_{j+1}, \ldots, \neg B_{n-1}$ and $\Sigma_2, \neg A_0, B_1, \ldots, B_j, \neg B_{j+1}, \ldots \neg B_{n-1}$, respectively. An application of MP_g now yields derivability of $(A_0 \Rightarrow B_0), \ldots, (A_j \Rightarrow B_j), \neg(A_{j+1} \Rightarrow B_{j+1}), \ldots, \neg(A_{n-1} \Rightarrow B_{n-1}), B_1, \ldots, B_j, \neg B_{j+1}, \ldots, \neg B_{n-1}$. Iterating the same schema, where MPEM_g is used instead of MP_g to eliminate occurrences of $\neg B_i$ for $i > j$ finally yields that $(A_0 \Rightarrow B_0), \ldots, (A_j \Rightarrow B_j), \neg(A_{j+1} \Rightarrow B_{j+1}), \ldots, \neg(A_{n-1} \Rightarrow B_{n-1}), \Gamma, \Delta$ is derivable in $\mathsf{GCKCEMMP}_g + \mathsf{Cut}(F, r_1, r_2)$.

To see that cuts between conclusions of CKCEM_g and MPEM_g can be eliminated, one uses the same reasoning as above, with MPEM_g and MP_g interchanged.

Proof of Theorem 31

Proof. We use dynamic programming as in [13]: we proceed in stages; at stage i, we compute provability of all sequents of length at most 2 and nesting depth of conditionals at most i consisting of subformulas of the target formula or their negations. As there are at most linearly many stages, it suffices to show that each stage can be performed in coNP. To this end, observe that proofs may be normalised to proceed as follows: first apply the propositional rules as long as possible, thus decomposing all formulas into literals of the form $A \Rightarrow B$ or $\neg(A \Rightarrow B)$ in the various branches of the proof. Then in each branch, the rule (CEM_g) is deterministically applied to maximal groups of conditionals or negated conditionals with equivalent left arguments; equivalence of the latter has been computed and memoised in the previous stages. This leads to proofs where all branching is universal, namely on rules with several premises (actually, only on the conjunction rule, as universal branching on (CEM_g) is immediately caught by memoising); there is no existential branching on selection of instances of rules to be applied. Thus, the search process is performed in coNP.

Proof of Theorem 32

Proof. Analogous to Theorem 31, with a slightly adapted proof strategy in between two applications of (CEM_g). To begin, we recall that, as laid out in the arguments leading up to the theorem, the rules (MP_g) and (MPEM_g) can be restricted to be applied at most once to every conditional or negated conditional in between two applications of (CEM_g). Thus, in the beginning of the (backwards) proof and after every application of (CEM_g), we apply the propositional rules and the rules (MP_g) and (MPEM_g) as long as possible, but obeying the mentioned restriction. The remaining argument is as before, noting that the proof depth remains polynomial.