Coalgebraic Correspondence Theory

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Abstract. We lay the foundations of a first-order correspondence theory for coalgebraic logics that makes the transition structure explicit in the first-order modelling. In particular, we prove a coalgebraic version of the van Benthem/Rosen theorem stating that both over arbitrary structures and over finite structures, coalgebraic modal logic is precisely the bisimulation invariant fragment of first-order logic.

Introduction

Viewing modal logic as a sub-language of first-order logic via the standard translation is the starting point of modal correspondence theory. One of the core results of this area, van Benthem's theorem [23], states that modal logic is precisely the bisimulation invariant fragment of first-order logic over relational structures. This result has been extended to finite structures by Rosen [19] (a first-first order formula is bisimulation invariant over finite structures iff it is logically equivalent to a modal formula over finite structures), and special frame classes have been considered in [5]. Results of this kind characterize the expressive power of modal logic – slightly reworded, they state that modal logic can express the same bisimulation-invariant properties as first-order logic.

Here, we extend these results to coalgebraic modal logic [16], thus making initial forays into *coalgebraic correspondence theory*. Coalgebraic modal logic is a framework for modal logics that parametrizes the semantics over the choice of a signature functor and a set of predicate liftings and serves as a generic framework capturing a wide range of modal logics from the literature, including e.g. classical and normal modal logics [3], graded and probabilistic modal logics [7,14,12], coalition logic [18], and conditional logics [3].

Our correspondence language is a multi-sorted first-order logic, inspired by the correspondence language for neighbourhood frames of [11]. It includes a dedicated sort to represent the (action of) the signature functor and thus provides a full model of coalgebras, and in fact of supporting Kripke frames of coalgebras in the sense of [22]. We follow the method of Rosen (and a related proof by Otto [15]) to prove that, under suitable assumptions, coalgebraic modal logic is, both over finite and over arbitrary structures, precisely the fragment of the coalgebraic correspondence language characterized by *invariance under behavioural equivalence.* As the method employed avoids compactness and saturation, which feature prominently in the original proof of van Benthem's theorem, we can deal

also with classes of coalgebras that fail to be first-order axiomatizable, which is a fairly typical phenomenon.

To show that a first-order formula that is invariant under behavioral equivalence can be characterized by a *finitary* formula, we have to assume that the underling singnature functor preserves finite sets. This covers Kripke and neighbourhood semantics, as well as the selection function semantics of conditional logic and a bounded version of graded modal logic, but excludes e.g. graded and probabilistic modal logic. In absence of this assumption, we do provide a characterization result for invariance under behavioural equivalence in terms of an extended, infinitary modal logic, together with a counter-example showing that equivalence to a finitary modal formula fails in general. This result applies to essentially all logics of interest, and indeed covers most of the way to the finitary result in terms of its proof, as the latter is an easy corollary to it in the case of finite signatures. One of the results we obtain as applications in this way is that every formula in a natural first order logic with counting quantifiers over multigraphs that is invariant under behavioural equivalence over finite structures is equivalent to a possibly infinitary formula *of bounded depth* in graded modal logic. Although similar results have previously been obtained over the class of *all* structures [6], our result seems to be the first Rosen-type result for graded modal logic over *finite* structures.

We note that the design of the correspondence language used as the setting for our results is a delicate affair: the translation of coalgebraic semantics, like already the translation of neighbourhood semantics used in [11], needs to include a sort of neighbourhoods, i.e. subsets of the state space. On the other hand, we need to avoid the full expressive power of monadic second order logic, in which the van Benthem/Rosen theorem fails to hold, as it contains the μ-calculus (which, in the standard relational case, is in fact its bisimulation-invariant fragment [13]). This forces us to adopt a Henkin-style interpretation of the neighbourhood sort by suitably constrained subsets of the full powerset. In our setup, the key to a suitable notion of model in this sense is the inclusion of explicit distinguished supports in the language; a particularly pleasant effect of this language extension is that it simultaneously acts as the key to enabling the use of Gaifman locality.

1   Coalgebra and Modal Logic

Throughout the paper, we fix a modal similarity type Λ consisting of modal operators with associated arities. As we will be considering models rather than frames, we express propositional variables as nullary modal operators. The set $\mathcal{F}(\Lambda)$ of Λ-*formulas* is then given by the grammar

$$\mathcal{F}(\Lambda) \ni \phi, \psi ::= \bot \mid \phi \wedge \psi \mid \neg\phi \mid \heartsuit(\phi_1, \ldots, \phi_n)$$

where $\heartsuit \in \Lambda$ is n-ary. We denote the language that admits arbitrary conjunctions of sets of formulas rather than just binary conjunctions by $\mathcal{F}_\infty(\Lambda)$. We write $\mathrm{rank}(\phi)$ for the maximal nesting depth of modal operators in the formula ϕ, defined formally as $\mathrm{rank}(\bot) = 0$, $\mathrm{rank}(\bigwedge \Phi) = \sup_{\phi \in \Phi} \mathrm{rank}(\phi)$, $\mathrm{rank}(\neg\phi) =$

rank(ϕ) and rank$(\heartsuit(\phi_1, \ldots, \phi_n)) = 1 + \max\{\text{rank}(\phi_1), \ldots, \text{rank}(\phi_n)\}$. Thus, the rank of a formula in $\mathcal{F}_\infty(\Lambda)$ may be infinite.

Formulas over Λ are interpreted over coalgebras with respect to a Λ-*structure* that consists of an endofunctor $T : \mathsf{Set} \to \mathsf{Set}$ on the category of sets, together with an assignment

$$[\![\heartsuit]\!] : \mathcal{Q}^n \to \mathcal{Q} \circ T$$

of natural transformations (the *predicate liftings*) where $\mathcal{Q} : \mathsf{Set}^{op} \to \mathsf{Set}$ is the contravariant powerset functor. The endofunctor T is called the *underlying endofunctor* of the structure, and we usually refer to the structure just in terms of its underlying endofunctor, leaving the assignments of predicate liftings implicit. We will implicitly assume that T is well-behaved in the following sense:

Assumption 1. We can assume w.l.o.g. that T preserves injective maps [1]. For convenience of notation, we will in fact sometimes assume that subset inclusions $X \hookrightarrow Y$ are mapped to subset inclusions $TX \hookrightarrow TY$. Moreover, we assume w.l.o.g. that T is non-trivial, i.e. $TX = \emptyset \implies X = \emptyset$ (otherwise, $TX = \emptyset$ for all X).

Using the above assumption, we can give a simple definition of *support*, which will play a role in our correspondence language:

Definition 2. A set $A \subseteq X$ is a *support* of $t \in TX$ if $t \in TA$

Note that we assume $TA \subseteq TX$. Support has played a role in various coalgebraic model constructions, see e.g. [22]. We keep the notion of support as broad as possible, and in particular do not insist on minimality, as the set of supports of $t \in TX$ does not necessarily have a smallest element with respect to subset inclusion [9].

Given a Λ-structure T, a T-*coalgebra* is a pair (C, γ) where C is a set (of states) and $\gamma : C \to TC$ is a (transition) function. We identify T-coalgebras (C, γ) with their carrier set C in case the transition function is clear from the context. The *semantics* $[\![\phi]\!]_C \subseteq C$ of a Λ-formula ϕ with respect to a T-coalgebra (C, γ) is given inductively by

$$[\![\heartsuit(\phi_1, \ldots, \phi_n)]\!]_C = \gamma^{-1} \circ [\![\heartsuit]\!]_C([\![\phi_1]\!]_C, \ldots, [\![\phi_n]\!]_C)$$

where $\heartsuit \in \Lambda$ is n-ary, together with the usual clauses for the propositional connectives. We write $(C, c) \models \phi$ if $c \in [\![\phi]\!]_C$.

Example 3. The following logics are covered by the coalgebraic approach.

1. Kripke models over a set P of propositional variables are triples (W, R, σ) where W is a set, $R \subseteq W \times W$ is a binary relation, and $\sigma : \mathsf{P} \to \mathcal{P}(W)$ is a valuation of propositional variables. It is easy to see that Kripke models are in 1-1 correspondence with T-coalgebras for $TX = \mathcal{P}(X) \times \mathcal{P}(\mathsf{P})$. The syntax of the modal logic K comes about via the similarity type $\Lambda = \{\Diamond\} \cup \mathsf{P}$ where $\Diamond$ is unary and each $p \in \mathsf{P}$ doubles as a nullary modality. The language $\mathcal{F}(\Lambda)$ is interpreted over T-coalgebras by virtue of the structure

$$[\![\Diamond]\!]_X(A) = \{(B, C) \in TX \mid B \cap A \neq \emptyset\} \qquad [\![p]\!]_X = (B, C) \in TX \mid p \in C\}$$

Clearly this semantics coincides with the standard textbook semantics of K [2].

2. The modal logic of neighbourhood frames (classical modal logic in [3]) arises via the same similarity type, but is interpreted over neighbourhood models, i.e. coalgebras for the functor $TX = \mathcal{PP}(X) \times \mathcal{P}(\mathsf{P})$ where again P is a set of propositional variables (for the categorically minded reader, we iterate the *contravariant* powerset functor). For a T-coalgebra (C, γ), we say that $A \subseteq C$ *is a neighbourhood of* $c \in C$ if $\gamma(c) = (N, B)$ where $A \in N$. The interpretation of propositional constants (nullary modalities) is as above and the semantics of classical modal logic arises via the lifting

$$[\![\Box]\!]_X(A) = \{(N, B) \in TX \mid A \in N\}$$

which again gives rise to the standard semantics.

3. Monotone modal logic has the same syntax as classical modal logic, but is interpreted over monotone neighbourhood models, i.e. coalgebras for the functor

$$TX = \{A \in \mathcal{PP}(X) \mid A \text{ upwards closed}\} \times \mathcal{P}(\mathsf{P})$$

where upwards closure refers to subset inclusion.

4. Conditional logic [3] has a binary modal operator $\Rightarrow$ that we write in infix notation. Conditional models over a set P of propositional variables come about as coalgebras for the functor

$$TX = \{f : \mathcal{P}(X) \to \mathcal{P}(X) \mid f \text{ a function}\} \times \mathcal{P}(\mathsf{P})$$

(again, the powerset on the left of the function space is contravariant) where propositional constants are interpreted as above and the lifting

$$[\![\Rightarrow]\!]_X(A, B) = \{(f, D) \in TX \mid f(A) \subseteq B\}$$

induces the standard semantics of conditional logic.

5. The similarity type of graded modal logic features, apart from propositional constants, an indexed collection of operators $\Diamond_k$ for $k \in \omega$. The intuitive reading of $\Diamond_k \phi$ is that ϕ holds in more than k successors. To retain naturality of predicate liftings, we slightly deviate from the traditional semantics [7] and interpret graded modal logic over coalgebras of the functor T (left) where we use the liftings $[\![\Diamond_k]\!]$ (right)

$$TX = \{f : X \to \omega\} \times \mathcal{P}(\mathsf{P}) \qquad [\![\Diamond_k]\!]_X(A) = \{(f, D) \in TX \mid \sum_{x \in A} f(x) > k\}$$

to interpret modal operators. In other words, we interpret graded modal logic over multigraphs [4] where the graded modalities refer to the weighted sum of successors. This semantics is equivalent to the standard Kripke semantics w.r.t. satisfiability of formulas, as multigraphs can be converted to Kripke frames by inserting the appropriate number of copies for each successor [20].

A variation of graded modal logic arises by limiting the overall (weighted) sum of successor states. If we consider the sub-functor

$$T_k X = \{f \in TX \mid \sum_{x \in X} f(x) \le k\} \times \mathcal{P}(\mathsf{P})$$

for some $k \geq 0$, we may describe k-bounded multigraphs as T-coalgebras, and interpret the sub-language that only features the modalities $\lozenge_i$ for $i < k$.

6. For probabilistic logics, we prefer to work with subprobabilities for technical reasons, where a subprobability distribution P on a set X is only required to assign measure ≤ 1 to the whole space. The similarity type of the *modal logic of sub-probabilities*, a variant of probabilistic modal logic [12], contains, apart from propositional variables, the modal operators M_p for rational $p \in [0,1] \cap \mathbb{Q}$. This language is interpreted over T-coalgebras where TX is the set of finitely supported subprobability distributions over X, that is,

$$TX = \{\mu : X \to [0,1] \mid \sum_{x \in X} \mu(x) \leq 1\} \times \mathcal{P}(\mathsf{P})$$

where the modalities M_p, read as "with probability of more than p", are interpreted via the liftings $[\![M_p]\!]_X(A) = \{(\mu, D) \in TX \mid \sum_{x \in A} \mu(x) > p\}$ which induces, up to the move to subprobabilities, the standard semantics.

We note that all similarity types except that of (unbounded) graded modal logic and the modal logic of subprobabilities are finite, provided that we only have finitely many propositional variables.

This paper characterizes the expressive power of $\mathcal{F}(\Lambda)$ as the fragment of first-order logic that is invariant under behavioural equivalence. The latter is most conveniently described in terms coalgebra homomorphisms. Given two T-coalgebras (C, γ) and (D, δ), a *morphism* between C and D is a function $f : C \to D$ that satisfies $\delta \circ f = Tf \circ \gamma$. Given two T-coalgebras (C, γ) and (D, δ), two states $(c, d) \in C \times D$ are called *behaviourally equivalent*, written $C, c \approx D, d$, if they can be identified by a morphism of T-coalgebras, that is, there are two morphisms $f : (C, \gamma) \to (E, \epsilon)$ and $g : (D, \delta) \to (E, \epsilon)$ into a T-coalgebra (E, ϵ) such that $f(c) = g(d)$. It is easy to see that $\mathcal{F}(\Lambda)$ is invariant under behavioural equivalence:

Lemma 4. *Let C, D be T-coalgebras and let $(c, d) \in C \times D$ be behaviourally equivalent. Then $c \models \phi$ iff $d \models \phi$ for all $\phi \in \mathcal{F}(\Lambda)$.*

In other words, modal formulas are invariant under behavioural equivalence. Our main theorem extends [23] to a coalgebraic setting and establishes that all first order formulas in a suitable correspondence language with this property are in fact equivalent to modal formulas. The proof follows Rosen [19] and Otto [15], and in particular makes use of the stratification of behavioural equivalence that explicitly accounts for the number of transition steps. From a coalgebraic perspective, this comes about by considering the projections of (states of) coalgebras into the so-called *terminal sequence* of the underlying endofunctor. The objects of the terminal sequence are given by $T_0 = 1$ for an arbitrary one-element set and $T_n = T(T_{n-1})$, and are connected by functions $p_n : T_{n+1} \to T_n$ where $p_0 : T_1 \to 1$ is uniquely determined and $p_n = Tp_{n-1}$. Every T-coalgebra (C, γ) defines a cone over the terminal sequence by $\gamma_0 : C \to 1$ and $\gamma_n = T\gamma_{n-1} \circ \gamma : C \to T_n$. Given two T-coalgebras C and D, we can now call a pair $(c, d) \in C \times D$ *n-step equivalent*, in symbols $C, c \approx_n D, d$, if $\gamma_n(c) = \delta_n(d)$. The following lemma relates n-step equivalence and behavioural equivalence:

Lemma 5. *Let C, D be T-coalgebras, and let $n \geq k \in \omega$. For $(c, d) \in C \times D$, $c \approx d$ implies that $c \approx_n d$ implies that $c \approx_k d$.*

The converse is true if modal operators distinguish "enough" successor states.

Definition 6. The Λ-structure T is *separating* if, for all sets X, every $t \in TX$ is uniquely determined by $\{(\heartsuit, A) \mid \heartsuit \in \Lambda \ n\text{-ary}, A \in \mathcal{P}(X)^n, t \in [\![\heartsuit]\!]_X(A)\}$.

Separation is sufficient to establish the Hennessy-Milner property for coalgebraic modal logics [17,21], and all the structures in Example 3 are indeed separating. In particular, we obtain characteristic formulas for n-step equivalence.

Lemma 7. *If the Λ-structure T is separating and (C, γ) is a T-coalgebra, every $\approx_k$-equivalence class is definable by a formula in $\mathcal{F}_\infty(\Lambda)$ of modal rank $\leq k$.*

2 From Coalgebraic Models to First-Order Structures

The characterization of (coalgebraic) modal logic as a fragment of first-order logic stands or falls with the first-order correspondence language under consideration. In general, one needs to balance expressivity of the correspondence language against the characterization results, and the value of our results increases with the expressive power of the correspondence language. The first-order correspondence language that we use here is inspired by [11] as it is multi-sorted and includes specific sorts for states and neighbourhoods. However, it also includes a third sort for *structured successors*, i.e. elements of the set TS where S is the state set. This is, to our taste, not only the most natural first-order modelling of coalgebras, but also strengthens our main result as it increases the expressivity of the correspondence language. Even more expressivity is owed to a slightly surprising feature, whose motivation is more technical in nature: one needs expressive means for the notion of support (Definition 2), which serves the dual purpose of restricting neighbourhoods (thus keeping the logic away from full monadic second-order logic) and on the other hand to avoid vacuity of Gaifman locality (see Remark 10 for details). The following non-essential assumption serves to simplify the presentation.

Assumption 8. We assume that $T\emptyset$ has a distinguished element $\perp_T$, and hence that every set TX also has a distinguished element $Ti(\perp_T)$, also denoted $\perp_T$, where $i : \emptyset \to X$ is inclusion. Moreover, we assume that $\perp_T \notin [\![\heartsuit]\!]_\emptyset(\emptyset, \ldots, \emptyset)$ for every k-ary operator $\heartsuit \in \Lambda$, and hence, by naturality, that $\perp_T \notin [\![\heartsuit]\!]_X(A_1, \ldots, A_k)$ for all sets X and all $A_1, \ldots, A_k \subseteq X$. This is mainly for the sake of readability, as it makes the definition of the standard translation more straightforward. In our running examples, $\perp_T$ can be taken to be

- $\perp_T = \emptyset \in \mathcal{P}(\emptyset)$ for the modal logic K (presented in terms of $\Diamond$);
- $\perp_T = \emptyset \in \mathcal{P}(\mathcal{P}(\emptyset))$ for classical and monotone modal logic;
- $\perp_T = \lambda A. \emptyset \in \mathcal{P}(\emptyset) \to \mathcal{P}(\emptyset)$ for conditional logic (presented in terms of $>$, defined by $\phi > \psi :\equiv \neg(\phi \Rightarrow \neg\psi)$).
- $\perp_T = \lambda x. 0$ for graded modal logic (presented in terms of the $\Diamond_k$);

– $\perp_T = \lambda x. 0$ for the modal logic of subprobabilities.

Definition 9. The *(coalgebraic) correspondence language* associated with the modal similarity type Λ is the first order language with equality $\mathcal{L}(\Lambda)$ over three sorts s, t, n of *states, successor structures,* and *neighbourhoods*, respectively, consisting of the sorted relation symbols

- tr $: s \times t$ (the coalgebraic transition structure)
- $\heartsuit : t \times n \times \cdots \times n$ (k copies of n) for all k-ary $\heartsuit \in \Lambda$ (the modal operators)
- $\in: s \times n$ (membership of points in neighbourhoods)
- supp $: t \times n$ (support, see Definition 2)

We notationally treat tr as a partial map, where the use of the term $\mathsf{tr}(x)$ for some x entails that $\mathsf{tr}(x)$ is defined. Given a first-order $\mathcal{L}(\Lambda)$-structure M, we denote the constituents of M by indexing as usual; e.g. M_s is the state set of M, and M_{tr} the successor relation. We say that M is *based on* a T-coalgebra (C, γ) if the following holds.

- $M_s = C$, $M_t \subseteq TC$ and $M_n \subseteq \mathcal{P}(C)$
- The relation M_{tr} is right-unique, and hence will be written as a partial map. It represents the transition structure γ with default value $\perp_T$; i.e. for each $c \in C$, $\gamma(c) = M_{\mathsf{tr}}(c)$ whenever $M_{\mathsf{tr}}(c)$ is defined, and $\gamma(c) = \perp_T$ otherwise.
- The relation M_{supp} is functional, and will also be written as a map. It picks a distinguished support (Definition 2) for every $\mu \in M_t$, i.e. $\mu \in T(M_{\mathsf{supp}}(\mu))$.
- The relations $M_\heartsuit$ represent the predicate liftings for every $\heartsuit \in \Lambda$ relative to the support, i.e. $M_\heartsuit = \{(\mu, A_1, \ldots, A_n) \in M_t \times \mathcal{P}(M_{\mathsf{supp}}(\mu))^n \mid \mu \in [\![\heartsuit]\!]_{M_{\mathsf{supp}}(\mu)}(A_1, \ldots, A_n)\}$ for $\heartsuit \in \Lambda$.
- $M_\in$ is elementhood: $M_\in = \{(s, A) \in C \times M_n \mid s \in A\}$.

We write $\mathsf{Mod}(\mathcal{L}(\Lambda))$ for the class of all $\mathcal{L}(\Lambda)$-structures that are based on some T-coalgebra, briefly referred to as T-*structures*. As every T-structure induces a uniquely defined T-coalgebra (C, γ) we occasionally regard T-structures as T-coalgebras, and in particular use notions such as behavioural equivalence also for T-structures. If M is a first-order structure for $\mathcal{L}(\Lambda)$ and $\phi(x) \in \mathcal{L}(\Lambda)$ is a formula with at most one free variable x of sort s, we write $M, m \models \phi(x)$ if the structure M with the free variable interpreted as m satisfies the formula $\phi(x)$, and $[\![\phi(x)]\!]_M$ is the set of all m such that $M, m \models \phi(x)$. We say that $\phi(x)$ is *invariant under behavioural equivalence* if $M, m \models \phi$ whenever $N, n \models \phi$ and $N, n \approx M, m$.

Our main interest in the present work is to establish results following van Benthem [23] and Rosen [19] which state that every first-order formula which is invariant under behavioural equivalence is equivalent to a modal formula, valid over the class of all structures and over the class of all finite structures, respectively. Occasionally we shall refer to results of the former kind as *van-Benthemtype* theorems, and to results of the latter kind as *Rosen-type* theorems.

Remark 10. Some explanations are in order concerning some aspects of the above definition. We first note that it is crucial that we do not require that the sort n of neighbourhoods is interpreted by the entire powerset of states. Otherwise, we would essentially arrive at monadic second-order logic, and hence invalidate the main theorem already for the case of the modal logic K as the bisimulation-invariant fragment of monadic second-order logic is the μ-calculus rather than the basic modal logic K [13]. Definition 9 restricts the interpretation of n only by the clause on the interpretation of the modal operators. Technically, this makes less formulas invariant under behavioural equivalence, as the interpretation of n may differ among behaviourally equivalent models. This is the first effect of support: without support, the interpretation of $\heartsuit \in \Lambda$ would need to be defined as something like $M_\heartsuit = \{(\mu, (A_1, \ldots, A_n) \in M_t \times \mathcal{P}(C)^n \mid \mu \in [\![\heartsuit]\!]_C(A_1, \ldots, A_n)\}$ which would constrain the interpretation of n much more strongly, and e.g. in the case of the modal logic K (with $\heartsuit = \Diamond$) would imply $A \in M_n$ for every $A \in \mathcal{P}(C)$ containing some state that has a predecessor.

The second technical point where support is needed is the following. The core of the proof of Rosen's theorem, as adapted below, is *locality*. In particular, we use Gaifman's theorem stating that every first-order formula is essentially local (see Section 3). Without support, however, locality becomes a void notion in many logics. E.g. in the extension of classical modal logic with necessitation, i.e. with an axiom $\Box\top$, any two points in the model would be connected by a path of length 3 (via the successor structure of the first point and the neighbourhood $\top$). As support has already played an important technical role in other contexts [22], and is also at the heart of our unravelling construction, we are beginning to believe it may be more than just a technical nuisance.

Finally, the purpose of the default value in the above definition of T-structure is to deal with substructures that arise by cutting off transitions after a fixed number of steps. In the setting of Kripke frames, this corresponds to all successors of a node x being lost in a substructure, so that x has the empty successor set, and hence fails to satisfy diamond formulas. This notion of cutting off transitions is made explicit by our default mechanism.

The correspondence language contains standard or natural correspondence languages when applied to the running examples, as illustrated next. In most cases, the generic language is even substantially more expressive than the 'natural' correspondence language, and van Benthem/Rosen-type results become *stronger* in the context of more expressive languages, as they apply to more formulas. Some of the examples moreover highlight the importance of support.

Example 11. 1. The coalgebraic correspondence language for K differs rather substantially from the standard first-order correspondence language, the language with a unary predicate p for every propositional variable p and a binary predicate R for the transition relation — it does not explicitly talk about the transition relation, and instead has types for successor sets and neighbourhoods (one of which could be dispensed with in this case). Crucially, the standard correspondence language can be embedded into the language used here, so that our characterization result established in Section 4 does reprove, and in fact

strengthen, the classical van Benthem/Rosen theorem. The embedding is defined by mapping atomic formulas xRy in the standard correspondence language to the formula $\exists A.\,(\mathsf{tr}(x)\Diamond A \wedge \forall z.\,(z \in A \leftrightarrow z = y))$.

2. In the case of classical modal logic with neighbourhood frame semantics, our correspondence language has a sort s of states, a sort t of sets of neighbourhoods, and a sort n of neighbourhoods, with the successor map tr which maps states to sets of neighbourhoods, the relation $\Box$ which represents the $\ni$ relation between sets of neighbourhoods and neighbourhoods, elementhood $\in$ between states and neighbourhoods, and additionally the support map supp from set of neighbourhoods to neighbourhoods. Superficially, this appears to be an extension of the correspondence language for neighbourhood frames used in [11], which has two sorts of states and neighbourhoods, respectively, corresponding to our sorts s and n, and two relations, corresponding to the composite $\mathsf{tr};\Box$ and to $\in$, respectively, in our setting. However, the two languages have a subtly different semantics in that in our language, the sets required to be present in the neighbourhood type are determined by the support, while in [11], the neighbourhood type contains precisely the image of the neighbourhood relation between states and neighbourhoods. At present, it is unclear whether one language can be embedded in the other so that the Benthem-type result of [11] remains independent of our characterization theorem result below (there is no correspondent in [11] to our Rosen-type result).

We emphasize that our results will apply without further ado to extensions of classical modal logic by rank-1 frame conditions, i.e. axioms where the nesting depth of modal operators is uniformly equal to 1. A simple example is the monotonicity axiom $\Box(a \wedge b) \rightarrow \Box a$, which axiomatizes monotone neighbourhood frames (Example 3.3), and it is unclear to what extent this is possible following [11]. In particular, one cannot interpret the type of neighbourhoods as the set of all sets that are a neighbourhood of some state, as the induced logic would be able to express some (if not all) μ-calculus formulas, such as $(\Box\top \wedge \neg\Box\neg\Box\bot) \rightarrow \mu X.p \wedge \Box X$: as soon as the antecedent $\Box\top \wedge \neg\Box\neg\Box\bot$ is satisfied, there is a state which has $\emptyset$ as a neighbourhood, so that neighbourhood type would contain *every* set of states by monotonicity, which allows us to define the fixoint above by quantification over neighbourhoods. As formulas of this type are invariant under behavioural equivalence but not equivalent to any modal formula, the analogue of the Benthem/Rosen theorem fails. In the approach presented here, the notion of support enables sufficiently small interpretations of the neighbourhood type and handles rank-1 frame conditions smoothly.

3. One natural correspondence language for graded modal logic, interpreted over multigraphs (thus recovering invariance under behavioural equivalence, which fails over the relational semantics) is an extension of first order logic with counting quantifiers where the counting is relative to local weights induced by the weighting of successors in a multigraph. This induces counting quantifiers $\exists^x_{>k} y.\,\phi$ read 'in the local weighting at x, there exist more than k y satisfying ϕ'. This language can be mapped into our language by a recursive translation $(\_)^t$, where the clause for a counting quantifier is $(\exists^x_{>k} y.\,\phi)^t = (\exists A :$

$n. \, \text{tr}(x) \Diamond_k A \wedge \forall y. \, (\phi)^t \leftrightarrow y \in A)$. Support is uncritical in this case (as already for item 1), as the first-order language with counting quantifiers does not contain a sort for neighbourhoods. The standard translation factors through the language of counting quantifiers via translations $(\_)^s_x$ taking $\Diamond_k \phi$ to $\exists^x_{>\,k} y. \, (\phi)^t_y$. As a consequence, the characterization results proved below apply *a forteriori* also to the language with counting quantifiers.

4. Similarly, the correspondence language for the modal logic of subprobabilities contains a sublanguage that speaks about locally determined weights of formulas: borrowing notation from Halperns first-order logic of so-called type-1 probability structures [10] (which we extend from a single, global probability distribution to Markov chains, i.e. local (sub-)probability distributions), we may write $w^x_y(\phi) \geq p$ to denote that the set of all y satisfying ϕ has, in the local distribution at x, probability at least p.

5. The correspondence language for conditional logic contains the following more natural language, consisting of three sorts s, t, n for states, selection functions, and neighbourhoods, respectively, unary state predicates for the propositional variables, and a ternary relation R of type $s \times n \times s$ giving for each neighbourhood a transition relation on states. As neighbourhoods are explicit, we need to retain the support function supp. This corresponds to the view of a selection function model as a multi-relational Kripke model where relations are indexed over propositions, i.e. a structure of the type $(X, (R_A \subseteq X \times X)_{A \subseteq X})$, where we retain information only about those R_A for which A is in the neighbourhood type. Here, it is again crucial that the neighbourhood type is required to contain only those sets that are contained in the support of some element – without the support, neighbourhoods would be the full powerset, thus affording the full expressive power of monadic second order logic, as every selection function on a set C is contained in $[\![\Rightarrow]\!](A, C)$ for every $A \in \mathcal{P}(C)$.

The translation of modal formulas to first-order logic takes the following form:

Definition 12. The *standard translation* $\text{ST}_x(\phi)$ of a modal formula $\phi \in \mathcal{F}(\Lambda)$ is a first-order formula with one free variable $x : s$ of sort s defined inductively by commutation with all boolean operators and

$$\text{ST}_x(\heartsuit(\phi_1, \dots, \phi_k)) = \exists A_1, \dots, A_k : n. \, (\text{tr}(x)\heartsuit(A_1, \dots, A_k) \wedge$$
$$\bigwedge_{i=1}^k \forall y : s. \, (y \in A_1 \leftrightarrow y \in \text{supp}(\text{tr}(x)) \wedge \text{ST}_y(\phi_1))).$$

The default value $\perp_T$ is compatible with the above definition: $\text{ST}_x(\heartsuit(\phi_1, \dots, \phi_k))$ is not satisfied in case $\text{tr}(x)$ is undefined, which agrees precisely with the behaviour of the default value $\perp_T$ inserted as the successor structure of x in this case. Correctness is a straightforward calculation:

Lemma 13. *Suppose $\phi = \text{ST}_x(\psi)$ for a modal formula $\psi \in \mathcal{F}(\Lambda)$. Let M be a first-order structure based on a coalgebra C. Then $[\![\phi]\!]_M = [\![\psi]\!]_C$. In particular, ϕ is invariant under behavioural equivalence.*

Remark 14. Unlike Rosen's proof [19], which we follow here, the original proof of van Benthem's characterization result, as well as e.g. the van-Benthem-type

result proved for neighbourhood structures in [11], rely on standard machinery from first-order logic, in particular compactness. There are at least two sources of non-compactness in the overall setup used here: one, of course, rests in the fact that we are aiming for a Rosen-type theorem over *finite models*; and the other is the functor T. Not only may T impose finite branching; e.g. in case T is the probability distribution functor, the set of formulas $\{\neg L_0 \neg p\} \cup \{M_{1-1/n}p \mid n \in \mathbb{N}\}$ for a propositional variable p is finitely satisfiable but not satisfiable, no matter what type of model (finite, infinite, finitely or infinitely branching) we consider.

3 Gaifman's Theorem and Coalgebraic Unravelling

We recall Gaifman's locality theorem [8], and derive a simple corollary that asserts locality of coproduct-invariant formulas (we claim no originality here). The basic idea is taken from [15], where the same statement is proved as Lemma 3.5 for at most binary relational structures. We apply a simpler if somewhat wholesale argument using Gaifman locality.

Definition 15. The *Gaifman graph* of a relational structure A is the graph whose nodes are the elements of A and contains and edge from a to b iff a and b occur together in one of the tuples in the interpretations of the relation symbols in A. (E.g., the Gaifman graph of a single-relation Kripke structure is just its symmetric closure.) *Gaifman distance* is graph distance in the Gaifman graph. For $l \geq 0$, the *l-neighbourhood* $N_l^A(a)$ of $a \in A$ is the induced substructure of A containing all points with Gaifman distance at most l from a. A first-order formula $\phi(x)$ with a single free variable x is *l-local* if for every relational structure A and every $a \in A$, $A, a \models \phi(x)$ iff $N_l^A(a), a \models \phi(x)$. Moreover, $\phi(x)$ is *Gaifman l-local* if for any two points $a, b \in A$ with isomorphic l-neighbourhoods, $A, a \models \phi(x)$ iff $A, b \models \phi(x)$.

The crucial difference between locality and Gaifman locality is that the latter admits global statements about the model A so that locality is stronger than Gaifman locality. We employ the following special case of Gaifman's theorem:

Theorem 16 (Gaifman [8]). *Every first-order formula $\phi(x)$ is Gaifman l-local for some $l \geq 0$, exponentially bounded in the quantifier rank of ϕ.*

Gaifman's theorem is usually formulated in single-sorted logic, but readily extends to multiple sorts, with the obvious definition of Gaifman distance, using the standard encoding of multiple sorts as unary predicates in single-sorted logic.

Definition 17. A formula $\phi(x)$ with a single free variable x is *invariant under coproducts* if for all relational structures A, B and all points $a \in A$, $A, a \models \phi(x)$ iff $A + B, a \models \phi(x)$, where $A + B$ is the coproduct (disjoint union) of A and B.

Corollary 18. *If $\phi(x)$ is invariant under coproducts, then $\phi(x)$ is l-local for some $l \geq 0$, exponentially bounded in the quantifier rank of $\phi(x)$.*

The previous does not immediately apply to our framework, as neighbourhoods in T-structures are not in general T-structures. However, the following lemma brings us back into the realm of T-structures, thanks to the existence of a default element $\perp_T \in T\emptyset$.

Lemma 19. *Let A be a T-structure, let a be a state in A, and let $k \geq 0$. Then $N_{3k}(a)$ is a T-structure.*

We proceed to develop some facts concerning (partial) tree unravellings of coalgebras, in generalization of corresponding techniques for Kripke frames, including a not entirely trivial coalgebraic generalization of the fact that on trees, behavioural equivalence is equivalent to bounded behavioural equivalence (Lemma 21). The basic notion underlying these concepts is the following.

Definition 20. Let A be a T-structure for the correspondence language. The *supporting Kripke frame* of A relates states $a, b \in A_s$ iff $b \in A_{\mathsf{supp}}(A_\gamma(a))$; i.e. its transition relation is $A_\gamma; A_{\mathsf{supp}}; A_\ni$ where $A_\ni$ is the inverse relation of $A_\in$. If this Kripke frame is a tree of depth l (with root a), i.e., is loop-free and every state is reachable from a by a unique path of length at most l, and moreover all leaves of this tree have the default successor structure (i.e. they do not have an R-successor) then we say that A (or (A, a)) is a *tree of depth l*.

Lemma 21. *Let A, B be T-structures with states $a \in A$, $b \in B$. If (A, a) and (B, b) are trees of depth at most l, then $A, a \approx B, b$ iff $A, a \approx_l B, b$.*

The core construction is described in the following lemma.

Lemma 22 (Unravelling). *Let A be a T-structure, let a be a state in A, and let $k \geq 0$. Then there exists a T-structure B and $b \in B$ such that $A, a \approx B, b$, and moreover $(N_{3k}^B(b), b)$ is a tree of depth at most k.*

Finally, we note that bounded behavioural equivalence is indeed local.

Lemma 23. *Let A be a T-coalgebra, let $a \in A$, and let $k \geq 0$. Then $A, a \approx_k N_{3k}^A(a), a$.*

4   A Coalgebraic van Benthem/Rosen Theorem

The core result proved in relational versions of the van Benthem/Rosen theorem is that every bisimulation-invariant formula can be expressed by a collection of modal formulas of bounded rank. In the relational case, this immediately implies equivalence to a single modal formula, as the set of modal formulas of a given maximal rank is finite up to logical equivalence. Coalgebraically, the situation turns out to be the same as long as the modal similarity type is finite. For infinite modal similarity types, the infinitary version of the van Benthem/Rosen theorem cannot be improved, as we demonstrate by means of a simple counterexample later. We thus tend to regard the infinitary version, stated next, as the most fundamental incarnation of the van Benthem/Rosen theorem. We emphasize

that the bound on the rank in the statement of the theorem is the core of the result – without it, the claim is a trivial consequence of the (coalgebraic) Hennessy-Milner property for infinitary languages [17]. As in [15], the theorem below has two readings, for finite and infinite models.

Theorem 24 (Coalgebraic van Benthem/Rosen theorem, infinitary version). *Let Λ be separating. A first-order formula $\phi(x)$ over T with a single free variable x is invariant under behavioural equivalence (over finite models) iff it is equivalent (over finite models) to a modal formula in $\mathcal{F}_\infty(\Lambda)$ with finite modal rank.*

The proof uses the Lemmas established in Section 3 in sequence. As announced, the finitary version of the theorem follows immediately for finite similarity types:

Corollary 25 (Coalgebraic van Benthem/Rosen theorem, finitary version). *Let Λ be finite and separating. Then a first-order formula $\phi(x)$ over T with a single free variable x is invariant under behavioural equivalence (over finite models) iff it is equivalent (over finite models) to a modal formula in $\mathcal{F}(\Lambda)$.*

For the logics introduced in Example 3, the situation is as follows.

Example 26. Theorem 24 applies to all logics of Example 3, and Corollary 25 applies to those logics that only have finitely many modalities, i.e. all of them except (unbounded) graded modal logic and probabilistic modal logic; we note that Corollary 25 does apply to our bounded version of graded modal logic. We emphasize that Theorem 24 does yield a characterization of the behavioural-equivalence-invariant fragment of a first-order logic with counting quantifiers; while a similar Benthem-type result is known [6], our result seems to be the first Rosen-type result (i.e. over finite structures) for graded modal logic.

As indicated above, a simple example shows that in the full correspondence language for an infinite modal similarity type, one can express properties which are invariant under behavioural equivalence but not expressible by a finitary modal formula, even in the standard coalgebraic modelling of Kripke models with infinitely many variables; in other words, the infinitary version of the van Benthem/Rosen theorem cannot be improved for infinite modal similarity types.

Example 27. Recall that standard Kripke models over the set P of variables are modelled by the functor $TX = \mathcal{P}(X) \times \mathcal{P}(\mathsf{P})$ (Example 3.1). Then the following formula is invariant under behavioural equivalence:

$$\exists y, z : s, Y, Z, A : n.$$
$$(\forall w : s. ((w \in Y \leftrightarrow w = y) \wedge (w \in Z \leftrightarrow w = z) \wedge w \in A)$$
$$\wedge \operatorname{tr}(x)\Diamond Y \wedge \operatorname{tr}(x)\Diamond Z \wedge \neg\operatorname{tr}(y)\Diamond A \wedge \neg\operatorname{tr}(z)\Diamond A \wedge \operatorname{tr}(y) \neq \operatorname{tr}(z))$$

This formula states that x has two successors y, z which are both deadlocks but disagree on the value of at least one propositional variable. This formula is evidently not equivalent to any finitary modal formula. However, it is expressible by the infinitary modal formula $\bigvee_{p\in\mathsf{P}}(\Diamond(p \wedge \neg\Diamond\top) \wedge \Diamond(\neg p \wedge \neg\Diamond\top))$, which has modal depth 2, thus illustrating Theorem 24.

Note that proofs of the Rosen theorem in a relational setting begin with a (trivial) reduction to finitely many variables, which is possible precisely because the standard correspondence language does not allow one to say that two states agree on all propositional variables. Of course, the example above depends heavily on the use of equality on t. We state the following nagging open question:

Problem 28. Let Λ be separating. Is every formula of the correspondence language that is invariant under behavioural equivalence and does not mention equality on t equivalent to a finitary modal formula?

We note that in the case of infinite collections of independent modal operators, such as infinitely many propositional variables, or boxes for infinitely many unrelated agents, the question is answered positively by a trivial reduction to the finite case. The problematic case are infinite collections of interdependent operators as, e.g., in graded modal logic.

5   Conclusions and Related Work

We have designed a correspondence language for coalgebraic modal logic, and proved two versions of the van Benthem/Rosen type theorem in this language: an infinitary version which applies to *every* coalgebraic modal logic, and shows that every formula which is invariant under behavioural equivalence is equivalent to an infinitary modal formula *of bounded depth*; and a finitary version, which is an easy corollary to the infinitary version, and shows that every formula which is invariant under behavioural equivalence is equivalent to a finitary modal formula but applies only to finite separating modal similarity types, a condition which implies in particular that the type functor preserves finite sets. The infinitary result yields e.g. that a formula in the first order logic of multigraphs with counting quantifiers is invariant under behavioural equivalence iff it is equivalent to a bounded-depth infinitary graded modal formula. The finitary result yields characterizations of conditional logic, classical modal logic, monotone modal logic, and a bounded version of graded modal logic as the invariant fragment under behavioural equivalence in the respective correspondence languages.

It remains an open problem to extend the finitary result to infinite modal similarity types; a simple example shows that this can work only for a restricted correspondence language that excludes equality on the type of successor structures. This would in particular imply van Benthem/Rosen theorems for graded and probabilistic modal logic. The former would complement a Benthem-type result for graded modal logic proved in [6] by a Rosen-type result (i.e. over finite structures). A further interesting direction for future investigation is to extend the ambient logic, in particular to obtain a coalgebraic analogue of the characterization of the modal μ-calculus as the bisimulation-invariant fragment of monadic second order logic due to Janin and Walukiewicz [13].

Acknowledgements

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Appendix: Omitted Proofs

Proof of Lemma 7

This is an easy adaptation of the corresponding proof in [17], and proceeds by induction on k. The case $k = 0$ is trivial. The $\approx_{k+1}$-equivalence class of $c \in C$ can be defined as

$$\bigwedge_{\substack{\heartsuit \in \Lambda \ n-\text{ary} \\ \gamma_k(c) \in [\![\heartsuit]\!]_{T_k}(A_1,\ldots,A_n)}} \heartsuit(\phi_{A_1},\ldots,\phi_{A_n}) \wedge \bigwedge_{\substack{\heartsuit \in \Lambda \ n-\text{ary} \\ \gamma_k(c) \notin [\![\heartsuit]\!]_{T_k}(A_1,\ldots,A_n)}} \neg\heartsuit(\phi_{A_1},\ldots,\phi_{A_n})$$

where ϕ_A is the (possibly infinite) disjunction of the formulas that define the k-equivalence classes of elements of $\gamma_k^{-1}(A)$. $\qquad\square$

Proof of Lemma 13

We proceed by induction on the structure of (the modal formula) ψ where $\psi = \heartsuit(\psi_1,\ldots,\psi_k)$ is the only case of interest. Let $M \in \mathsf{Mod}(\mathcal{L}(\Lambda))$, let $c \in C$, and denote the structure map of C by γ. We consider only the case that $M_{\mathsf{tr}}(c)$ is defined, having dealt with the other case in the previous explanations. We reason as follows: $C, c \models \phi$ iff $\gamma(c) \in [\![\heartsuit]\!]_C([\![\psi_1]\!]_C,\ldots,[\![\psi_n]\!]_C)$ iff, by naturality of $[\![\heartsuit]\!]$ and because $\gamma(c) \in M_{\mathsf{supp}}(\gamma(c))$, $\gamma(c) \in [\![\heartsuit]\!]_{M_{\mathsf{supp}}(\gamma(c))}(A_1,\ldots,A_k)$ with $A_i = [\![\psi_i]\!]_C \cap M_{\mathsf{supp}}(\gamma(c)) = [\![\psi_i]\!]_M \cap M_{\mathsf{supp}}(\gamma(c))$ for all i (where the last equality holds by induction) iff, as M is based on C, $(\gamma(c), A_1,\ldots,A_k) \in M_\heartsuit$ iff $M, c \models \mathsf{ST}_x(\heartsuit(\psi_1,\ldots,\psi_n))$. $\qquad\square$

Proof of Corollary 18

By Gaifman's theorem, $\phi(x)$ is Gaifman l-local for some $l \geq 0$; we prove that $\phi(x)$ is indeed l-local. Thus, let A be a relational structure, let $a \in A$, and let $B = N_l(a)$. We have to prove that $A, a \models \phi(x)$ iff $B, a \models \phi(x)$. Let $C = A + B$, with the left hand copy of a denoted a_1 and the right hand copy denoted a_2. Then $A, a \models \phi(x)$ iff $A + B, a_1 \models \phi(x)$ (by invariance under coproducts) iff $A + B, a_2 \models \phi(x)$ (by Gaifman l-locality) iff $B, a_2 \models \phi(x)$ (by invariance under coproducts). $\qquad\square$

Proof of Lemma 19

Let A be based on a coalgebra (C, γ). We define a coalgebra (C_0, γ_0) on which $N_{3k}(a)$ is based, as follows. The set of states is $C_0 = (N_{3k}(a))_s$. For $x \in C_0$, we put $\gamma_0(x) = (N_{3k}(a))_{\mathsf{tr}}(x)$ if this value is defined, and $\gamma_0(x) = \perp_T$ otherwise. We have to check that the conditions relating to the support and the interpretation of the relations $\heartsuit$ for the modal operators $\heartsuit \in \Lambda$ hold for $N_{3k}(a)$. This follows from the fact that the length of a Gaifman path between two states is always a multiple of 3, so that as soon as $(N_{3k}(a))_{\mathsf{tr}}(x)$ is defined for some x, all entities relevant for these conditions are retained in $N_{3k}(a)$. $\qquad\square$

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Proof of Lemma 21

'Only if' is trivial. To prove 'if', let α and β denote the coalgebra structure maps of A and B, respectively. Similarly as in [16], let $f^0 : 1 \to T1$ be the map induced by the distinguished element $\perp_T$ of $T1$, and put $f^n = T^n f^0$, a T-coalgebra on $T^n 1$. We shall show that the fact that A, a and B, b are trees of depth at most l implies that

$$f^l \alpha_l = \alpha_{l+1}, \qquad (*)$$

which means that $\alpha_l : A \to (T^n 1, f^n)$ is a T-coalgebra morphism as $\alpha_{l+1} = T\alpha_l\alpha$; similarly for β_l. As $\alpha_l(a) = \beta_l(b)$ by assumption, this implies the claim.

It remains to prove $(*)$. We generalize to $f^k \alpha_k = \alpha_{k+1}$ for all $k \geq l$, and proceed by induction over l.

$l = 0$: We conduct a further induction over k. For $k = 0$, the underlying tree of A, a consists only of the root a, and we have

$$f^0 \alpha_0(a) = \perp_T = T!\alpha(a)$$

where the second equality follows from the fact that $\alpha(a) = \perp_T$ and Tf preserves $\perp_T$. For $k > 0$, and assuming the claim for $k - 1$, we calculate

$$f^k \alpha_k = T(f^{k-1})T\alpha_{k-1}\alpha = T(f^{k-1}\alpha_{k-1})\alpha = T(\alpha_k)\alpha = \alpha_{k+1}$$

where we apply the inductive hypothesis in the second-to-last step.

$l - 1 \to l$: By the inductive hypothesis, the claim holds at all nodes of the tree below the root node a (which have tree depth at most $l - 1$). Thus, we have to show only that $f^k(\alpha_k(a)) = \alpha_{k+1}(a)$ for $k \geq l$. As above, we calculate

$$f^k \alpha_k(a) = T(f^{k-1})T\alpha_{k-1}\alpha(a) = T(f^{k-1}\alpha_{k-1})\alpha(a) = T(\alpha_k)\alpha = \alpha_{k+1}(a)$$

where we can apply the inductive hypothesis in the second-to-last step because $\alpha(a) \in TS$, where $S \subseteq X$ is the set of children of a in the underlying tree of A. $\qquad\square$

Proof of Lemma 22

Let A be based on a coalgebra $\gamma : C \to TC$. We construct a coalgebra D as the disjoint union $D = \coprod_{i=0}^{k} \{a\} \times C^i + C$ where we write in_l and in_r for the left (resp. right) injection. The idea here is that the left-hand summand is a tree-shaped unfolding of C up to depth k, and the right-hand summand C represents the original coalgebra structure, into which every path runs after more than k steps. Formally, the transition structure $\beta : D \to TD$ is defined by case analysis:

$$\beta(\mathsf{in}_r(\bar{a})) = T\mathsf{in}_r(\gamma(\bar{a}))$$
$$\beta(\mathsf{in}_l(a_0, a_1, \ldots, a_k)) = T\mathsf{in}_r(\gamma(a_k)) \qquad (a_0 = a)$$
$$\beta(\mathsf{in}_l(a_0, a_1, \ldots, a_j)) = Tf(\gamma(a_j)) \qquad (a_0 = a)$$

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where in last clause, $0 \leq j < k$ and $f : C \rightarrow D$ is defined by $f(\bar{a}) = \mathsf{in}_l(a, a_1, \ldots, a_j, \bar{a})$. We have a T-structure B based on D, with the support being an appropriate copy of C in each case above. We put $b = \mathsf{in}_r(a)$.

This turns the projection $\pi : D \rightarrow C$ defined by $\pi(\mathsf{in}_r(a)) = a$ and $\pi(\mathsf{in}_l(a_0, a_1, \ldots, a_i)) = a_i$ $(a_0 = a)$ into a coalgebra morphism, as

$$T\pi \circ \beta(\mathsf{in}_r(\bar{a})) = T\pi \circ T\mathsf{in}_r(\gamma(\bar{a})) = \gamma(\bar{a}) = \gamma \circ \pi(\mathsf{in}_r(\bar{a}))$$
$$T\pi \circ \beta(\mathsf{in}_l(a_0, \ldots a_k)) = T\pi \circ T\mathsf{in}_r(\gamma(a_k)) = \gamma(a_k) = \gamma \circ \pi(a_0, \ldots a_k)$$
$$T\pi \circ \beta(\mathsf{in}_l(a_0, \ldots, a_j)) = T\pi \circ Tf(\gamma(a_j)) = \gamma(a_j) = \gamma \circ \pi(\mathsf{in}_l(a_0, \ldots, a_j))$$

so that $C, a \approx B, b$. By construction, $(N_{3k}^B(b), b)$ is a tree of depth at most k (in fact, exactly k). $\square$

Proof of Lemma 23

Let α denote the transition structure of A. One shows by induction over $l \geq 0$ that for all l, $\alpha_l(a) \in T^l 1$ depends only on states reachable from a in l steps in the supporting Kripke frame and on the successor structures of states reachable in $l - 1$ steps. Up to $l = k$, all such elements are retained in $N_{3k}^A(a)$. $\square$

Proof of Theorem 24

To prove the non-trivial direction, let $\phi(x)$ be invariant under behavioural equivalence. The pattern of the proof is as in [15]: To begin, $\phi(x)$ is, in particular, invariant under coproducts (since coproduct injections are coalgebra morphisms), and therefore l-local for some $l \geq 0$ by Corollary 18. We can assume $l = 3k$ for some k.

In the next step, we prove that $\phi(x)$ is even $\approx_k$-invariant. The claim of the theorem then follows by Lemma 7.

Thus let $A, a \approx_k B, b$; we have to prove that $A, a \models \phi(x)$ iff $B, b \models \phi(x)$. By l-locality, it suffices to prove that $N_l^A(a), a \models \phi(x)$ iff $N_l^B(b), b \models \phi(x)$. By invariance under behavioural equivalence, this follows if $N_l^A(a), a \approx N_l^B(b), b$. By the unravelling lemma (Lemma 22), we may assume that $(N_l^A(a), a)$ and $(N_l^B(b), b)$ are trees of depth at most k. The claim then follows by Lemma 21 once we show that $N_l^A(a), a \approx_k N_l^B(b), b$, which however follows from $A, a \approx_k B, b$ and Lemma 23. $\square$