

Powermonads and Tensors of Unranked Effects

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Abstract—In semantics and in programming practice, algebraic concepts such as monads or, essentially equivalently, (large) Lawvere theories are a well-established tool for modelling generic side-effects. An important issue in this context are combination mechanisms for such algebraic effects, which allow for the modular design of programming languages and verification logics. The most basic combination operators are sum and tensor: while the sum of effects is just their non-interacting union, the tensor imposes commutation of effects. However, for effects with unbounded arities, these combinations need not in general exist. Here, we introduce the class of uniform effects, which includes unbounded nondeterminism and continuations, and prove that the tensor does always exist if one of the component effects is uniform, thus in particular improving on previous results on tensoring with continuations. We then treat the case of nondeterminism in more detail, and give an order-theoretic characterization of effects for which tensoring with nondeterminism is conservative, thus enabling nondeterministic arguments such as a generic version of the Fischer-Ladner encoding of control operators.

I. INTRODUCTION

Both in actual programming languages and in their semantics and meta-theory, one encounters a wide variety of phenomena that can be subsumed under a broadly understood notion of side-effect, such as various forms of state, input/output, resumptions, backtracking, nondeterminism, continuations, and many more. This proliferation of effects motivates the search for generic frameworks that encapsulate the exact nature of side-effects and support abstract formulations of programs (such as Haskell’s generic while-loop), semantic principles, and program logics. A fairly well-established abstraction of this kind is the modelling of side-effects as monads, following seminal work by Moggi [1]; this principle is widely used in programming language semantics (e.g. [2]–[5]) and moreover underlies the incorporation of side-effects in the functional programming language Haskell [6]. Besides supporting generic results that can be instantiated to particular effects at little or no cost, monads allow for a clear delineation of the scope of effects [7]. A more recent development is the advancement of Lawvere theories [8] for the generic modelling of effects, thus emphasizing their algebraic nature [9].

One advantage of these approaches is that they provide for a *modular* semantics of effects. It has been observed that many effects, such as state, exceptions, and continuations, induce so-called *monad transformers* that can be seen as adding the respective effect to a given set of effects [10], [11]; again, the notion of monad transformer plays a central role in Haskell. More recently, it has been shown that many monad transformers arise from binary combination operators that join effects in a prescribed way. The most important among these constructions are the *sum* of effects, which corresponds simply to the disjoint union of algebraic theories, and the *tensor*, which additionally imposes a commutation condition [12], [13]. E.g., the exception monad transformer is summation with the exception monad, and the state monad transformer is tensoring with the state monad [14]. These combination methods are often mixed; e.g. [15] uses both sums and tensors of nondeterminism with other effects. (Previous work on the specific combination of unbounded nondeterminism and probabilistic choice uses a different form of interaction than imposed by the tensor [16], [17].)

One of the problems that arise with sum and tensor in the context of *large* Lawvere theories, i.e. theories that can be *unranked* in the sense that their operations have unbounded arities, such as unbounded nondeterminism or continuations, is that for reasons of size, the combined theories need not exist in general; e.g. we show in recent work [18] that tensors of unranked theories with the theory of lists may fail to exist. In the present work, we introduce the notion of *uniform* theory, and prove that the tensor of two large Lawvere theories always exists if one of them is uniform. The class of uniform theories includes several variants of nondeterminism (e.g. unbounded and countable, but not finite) as well as, somewhat surprisingly, continuations; thus, our existence result improves a previous result stating that the tensor of any *ranked* theory with continuations always exists [12].

One may read this result as yielding a number of new monad transformers; we are particularly interested in nondeterminism monad transformers, which we dub *powermonads*. This leads us to a second problem associated specifically with the tensor: since the tensor imposes a complex algebraic interaction between the component effects, it cannot in general be expected to be *conservative* in the sense that the components embed into the tensor.

To deal with this issue in the special case of nondeterminism, we focus on *bounded* theories L , which come with a natural approximation ordering. We begin by giving a simplified construction of tensoring with nondeterminism, which is informed by but technically independent of the general existence result (and, e.g., applies also to tensoring finitary theories with finite nondeterminism, although the latter fails to be uniform): morphisms in the tensor of L with nondeterminism are sets of L -morphisms modulo *rectangular equivalence*, a comparatively simple equivalence that forces uniqueness of tupling morphisms. From there, we obtain a more order-theoretic description of the tensor in terms of *closed* sets of L -morphisms, which leads to a simple characterization of theories for which tensoring with nondeterminism is order-theoretically conservative.

The main reason for our interest in tensoring with nondeterminism is that it yields exactly the free extension of a given theory to a *completely additive* theory, i.e. one that is enriched over complete join semi-lattices; this amounts to having choice operators that distribute over sequential composition on both sides (hence providing a trace-based rather than a bisimulation-based perspective). Thus, whenever a theory L can be conservatively tensored with nondeterminism, one can conduct equational and order-theoretic proofs in it pretending that L is completely additive. E.g., one can use the well-known translation of imperative constructs [19]

$$\begin{aligned} \text{if } b \text{ then } p \text{ else } q &:= b?; p + (-b)?; q \\ \text{while } b \text{ do } p &:= (b?; p)^*; (-b)? \end{aligned}$$

that we dub the Fischer-Ladner encoding *generically*, i.e. for any effect satisfying our conservativity conditions. Besides simplifying the reasoning, this uncovers the nondeterministic flavour of imperative branching [20].

The material is organized as follows. We recall basic facts on monads and Lawvere theories in Section II. In Section III, we review tensor products, and proceed directly to the main existence result for tensors with uniform theories. We discuss additive theories in Section IV, and present our results on conservativity of tensoring with nondeterminism in Section V.

II. LARGE LAWVERE THEORIES AND MONADS

In a nutshell, the principle of monadic encapsulation of side-effects originally due to Moggi [1] and subsequently introduced into the functional programming language Haskell as the principal means of dealing with impure features [6] consists in moving the side effect from the function arrow into the result type of a function: a side-effecting function $X \rightarrow Y$ becomes a pure function $X \rightarrow TY$, where TY is a type of side-effecting computations over Y ; the base example is $TY = S \rightarrow (S \times Y)$ for a fixed set S of states, so that functions $X \rightarrow TY$ are functions that may read and update a global state (more examples will be given later). Formally, a *monad* on the category of sets, presented as

a *Kleisli triple* $\mathbb{T} = (T, \eta, _*)$, consists of a function T mapping sets X (of values) to sets TX (of computations), a family of functions $\eta_X : X \rightarrow TX$, and a map assigning to every function $f : X \rightarrow TY$ a function $f^* : TX \rightarrow TY$ that lifts f from X to computations over X . These data are subject to the equations $\eta^* = \text{id}$, $f^*\eta = f$, $(f^*g)^* = f^*g^*$, which ensure that the *Kleisli category* of \mathbb{T} , which has sets as objects and maps $X \rightarrow TY$ as morphisms, is actually a category, with identities $\eta : X \rightarrow TY$ and composition f^*g . On **Set**, all monads are *strong*, i.e. equipped with a natural transformation $X \times TY \rightarrow T(X \times Y)$ satisfying a number of coherence conditions [1].

Monads were originally intended as abstract presentations of algebraic theories, with TX abstracting the free algebra over X , i.e. terms over X modulo provable equality. It has been shown that the algebraic view of monads gives rise to computationally natural operations for effects; e.g. the state monad (with state set $S = V^L$ for sets V of values and L of locations) can be algebraically presented in terms of operations *lookup* and *update* [9]. Categorically, this shift of viewpoint amounts to generating monads from Lawvere theories. To cover unranked theories, we use the notion of large Lawvere theory [21], introduced into the theory of generic effects in [12]. Generally, we denote hom-sets of a category \mathbf{C} in the form $\mathbf{C}(A, B)$.

Definition 1 (Large Lawvere theory). A *large Lawvere theory* is given by a locally small category L with small products, together with a strict product preserving identity-on-objects functor $I : \mathbf{Set}^{op} \rightarrow L$. We call I the *indexing functor*, and we denote $I f$ by $[f]$ for a map f . A morphism of large Lawvere theories $L_1 \rightarrow L_2$ is a functor $L_1 \rightarrow L_2$ that commutes with the indexing functors (and hence preserves small products). A *model* of a large Lawvere theory L in a category \mathbf{C} with small products is a small product preserving functor $L \rightarrow \mathbf{C}$.

The algebraic intuition behind these definitions is that the objects of a large Lawvere theory are sets n, m, k, \dots of variables, and morphisms $n \rightarrow m$ are m -tuples of terms over n , or substitutions from m into terms over n . The indexing functor prescribes the effect of rearranging variables in terms. The notion of model recalled above implies that Lawvere theories provide a representation of effects that is independent of the base category \mathbf{C} , and given enough structure on \mathbf{C} , a Lawvere theory will induce a monad on \mathbf{C} . E.g., in categories of domains, the theory of finite non-blocking nondeterminism (Example 2.2 below) induces precisely the Plotkin powerdomain monad (while the Hoare and Smyth powerdomains require enriched Lawvere theories) [22].

It is well-known that large Lawvere theories and strong monads on **Set** form equivalent (overlarge) categories [12], [21]. The equivalence maps a large Lawvere theory L to the monad $T_L X = L(X, 1)$ (we elide the full description),

and a monad T to the dual of its Kleisli category. We therefore largely drop the distinction between monads and large Lawvere theories, and freely transfer concepts and examples from one setting to the other; occasionally we leave the choice open by just using the term *effect*.

We say that a large Lawvere theory L is *ranked* if it can be presented by operations (and equations) of arity less than κ for some cardinal κ ; otherwise, L is *unranked*. Categorically, L having rank κ amounts to preservation of κ -directed colimits by the induced monad. If L has rank κ , then L is determined by its full subcategory spanned by the sets of cardinality less than κ . If L has rank ω , we say that L is *finitary*.

Example 2. 1) *Global state*: as stated initially, $TX = S \rightarrow (S \times X)$ is a monad (for this and other standard examples, we omit the description of the remaining data), the well-known *state monad*. A variant is the *partial state monad* $TX = S \rightarrow (S \times X)_\perp$, where X_\perp extends X by a fresh element \perp representing non-termination. (This induces a relational model of non-termination in the spirit of PDL and related formalisms; a domain-theoretic treatment of non-termination requires a domain-enriched Lawvere theory in which \perp is explicitly a bottom element).

2) *Nondeterminism*: the *unranked* large Lawvere theory $L_{\mathcal{P}}$ for nondeterminism arises from the powerset monad \mathcal{P} . It has m -tuples of subsets of n as morphisms $n \rightarrow m$. Variants arise on the one hand by restricting to nonempty subsets, thus ruling out non-termination, and on the other hand by bounding the cardinality of subsets. We denote nonemptiness by a superscript $*$, and cardinality bounds by subscripts. E.g., the large Lawvere theory $L_{\mathcal{P}_n^*}$ describes finite non-blocking nondeterminism; its morphisms $n \rightarrow m$ are m -tuples of nonempty finite subsets of n . Yet another variant arises by replacing sets with multisets, i.e. maps $X \rightarrow (\mathbb{N} \cup \{\infty\})$, thus modelling weighted nondeterminism [23] as a large Lawvere theory L_{mult} .

3) *Continuations*: The continuation monad maps a set X to the set $(X \rightarrow R) \rightarrow R$, for a fixed set R of results. The corresponding *unranked* large Lawvere theory L_{cont}^R has maps $m \rightarrow ((n \rightarrow R) \rightarrow R)$ as morphisms $n \rightarrow m$.

4) *Input/Output*: For a given set I of input symbols, the Lawvere theory L_I for input is generated by a single I -ary operation *in*; it is an *absolutely free* theory, i.e. has no equations. Similarly, given a set O of output symbols, the Lawvere theory L_O for output is generated by unary operations *out_o* for $o \in O$.

Further effects that fit the algebraic framework are exceptions ($TX = X + E$), resumptions ($RX = \mu Y. T(X + Y)$) for a given base effect T) and many more.

Notation 3. Let L be a large Lawvere theory. For an object n of L and $i \in n$, we let κ_i denote the map $1 \rightarrow n$ that picks i . Thus, the κ_i induce product projections $[\kappa_i] : n \rightarrow 1$

in L . Given two sets n and m , their **Set**-product $n \times m$ can be viewed as the sum of m copies of n in **Set**, and hence as the m -th power of n in L . This induces for every $f : n \rightarrow k$ in L the morphisms $f \otimes m : n \times m \rightarrow k \times m$ and $m \otimes f : m \times n \rightarrow m \times k$.

A convenient way of denoting generic computations is the so-called *computational metalanguage* [1], which has found its way into functional programming in the shape of Haskell’s *do*-notation. We briefly outline the version of the metalanguage we use below.

The metalanguage serves to denote morphisms in the underlying category of a given monad, using the monadic structure; since large Lawvere theories correspond to monads on **Set**, the metalanguage just denotes maps in our setting. We let a signature Σ consist of a set \mathcal{B} of *base types*, to be interpreted as sets, and a collection of typed *function symbols* to be interpreted as functions. Here, we assume that *types* $A, B \in \mathcal{T}$ are generated from the base types by the grammar

$$A, B ::= 1 \mid A \in \mathcal{B} \mid A + B \mid A \times B \mid TA$$

where $+$ and \times are interpreted as set theoretic sum and product, respectively, 1 is a singleton set, and T is application of the given monad. We then have standard formation rules for terms-in-context $\Gamma \triangleright t : A$, read ‘term t has type A in context Γ ’, where a *context* is a list $\Gamma = (x_1 : A_1, \dots, x_n : A_n)$ of typed variables (later, contexts will mostly be omitted):

$$\frac{x : A \in \Gamma}{\Gamma \triangleright x : A} \quad \frac{f : A \rightarrow B \in \Sigma \quad \Gamma \triangleright t : A}{\Gamma \triangleright f(t) : B} \quad \frac{}{\Gamma \triangleright * : 1}$$

$$\frac{\Gamma \triangleright t : A \quad \Gamma \triangleright u : B}{\Gamma \triangleright \langle t, u \rangle : A \times B} \quad \frac{\Gamma \triangleright t : A \times B}{\Gamma \triangleright \text{fst } t : B} \quad \frac{\Gamma \triangleright t : A \times B}{\Gamma \triangleright \text{snd } t : B}$$

$$\frac{\Gamma \triangleright s : A + B \quad \Gamma, x : A \triangleright t : C \quad \Gamma, y : B \triangleright u : C}{\Gamma \triangleright \text{case } s \text{ of } \text{inl } x \mapsto t; \text{inr } y \mapsto u : C}$$

$$\frac{\Gamma \triangleright t : A}{\Gamma \triangleright \text{inl } t : A + B} \quad \frac{\Gamma \triangleright t : B}{\Gamma \triangleright \text{inr } t : A + B}$$

This syntax supports, e.g., the standard encoding of the if-operator as

$$\text{if } b \text{ then } p \text{ else } q = \text{case } b \text{ of } \text{inl } * \mapsto p; \text{inr } * \mapsto q,$$

for $b : 2$, where $2 = 1 + 1$. Beyond this, we have monadic term constructors

$$\frac{\Gamma \triangleright t : A}{\Gamma \triangleright \text{ret } t : TA} \quad \frac{\Gamma \triangleright p : TA \quad \Gamma, x : A \triangleright q : TB}{\Gamma \triangleright \text{do } x \leftarrow p; q : TB}$$

called *return* and *binding*, respectively. Return is interpreted by the unit η of the monad, and can be thought of as returning a value. A binding $\text{do } x \leftarrow p; q$ executes p , binds its result to x , and then executes q , which may use x (if not, mention of x may be omitted). It is interpreted using Kleisli composition and strength, where the latter serves to

propagate the context Γ [1]. In consequence, one has the *monad laws*

$$\begin{aligned} \text{do } x \leftarrow p; \text{ret } x = p & & \text{do } x \leftarrow \text{ret } a; p = p[a/x] \\ \text{do } x \leftarrow (\text{do } y \leftarrow p; q); r & = \text{do } x \leftarrow p; y \leftarrow q; r \end{aligned}$$

Terms of a type TA are called *programs*.

III. TENSORS OF LARGE LAWVERE THEORIES

One of the key benefits of the monadic modelling of effects is that it allows for a modular treatment, where effects are combined from basic building blocks according to the demands of the programming task at hand. In current programming practice (specifically in Haskell [24]), this is typically achieved by generalizing a given effect to a *monad transformer* [1], [11], i.e. a function that maps monads to monads, in the process extending them with a given effect. For instance, the *state monad transformer* ST for a given set S of states maps a given monad T to the monad $ST(T)$ with $ST(T)(X) = S \rightarrow T(S \times X)$. Monad transformers are very general, but do not support a great deal of meta-theoretic results, as no further properties are imposed on them; e.g., they need not be functorial. It has been shown in [13] that many monad transformers arise from a few basic binary operations on Lawvere theories (equivalently on monads). E.g., the exception monad transformer, which maps a monad T to the monad $T(_ + E)$ for a fixed set E of exceptions, is just summation with $_ + E$; expressed in terms of large Lawvere theories, the sum $L_1 + L_2$ of two effects L_1, L_2 is simply the disjoint union of the associated theories, i.e. is universal w.r.t. having morphisms $L_1 \rightarrow L_1 + L_2 \leftarrow L_2$. Another important operation is the *tensor* which additionally imposes a strong form of interaction between the component theories in the form of a commutation law.

Definition 4 (Tensor). [12] The *tensor* $L_1 \otimes L_2$ of large Lawvere theories L_1, L_2 is the large Lawvere theory which is universal w.r.t. having *commuting* morphisms $L_1 \rightarrow L_1 \otimes L_2 \leftarrow L_2$ (elided in the notation), if such a universal theory exists. Here, commutation is satisfaction of the *tensor law*, i.e. given $f_1 : n_1 \rightarrow m_1$ in L_1 and $f_2 : n_2 \rightarrow m_2$ in L_2 we demand commutativity of the diagram

$$\begin{array}{ccc} n_1 \times n_2 & \xrightarrow{n_1 \otimes f_2} & n_1 \times m_2 \\ f_1 \otimes n_2 \downarrow & & \downarrow f_1 \otimes m_2 \\ m_1 \times n_2 & \xrightarrow{m_1 \otimes f_2} & m_1 \times m_2. \end{array}$$

By the equivalence between large Lawvere theories and monads, this induces also a notion of *tensor of monads* [12]. The computational meaning of the commutation condition becomes clearer in the computational metalanguage: if we extended the metalanguage with subtypes $T_i A$ of TA interpreted using the component monads T_1, T_2 of the tensor

$T = T_1 \otimes T_2$, it amounts to the equality

$$\begin{aligned} \text{do } x_1 \leftarrow p_1; x_2 \leftarrow p_2; \text{ret}(x_1, x_2) = \\ \text{do } x_2 \leftarrow p_2; x_1 \leftarrow p_1; \text{ret}(x_1, x_2) \end{aligned}$$

in context Γ_1, Γ_2 , where $\Gamma_i \triangleright p_i : T_i A_i$ for $i = 1, 2$; i.e. programs having only effects from T_1 do not interfere with programs having only effects from T_2 .

Example 5. [14] Tensoring with the state monad $TX = S \rightarrow (S \times X)$ yields exactly the standard state monad transformer (in particular, tensors with T always exist).

Sum and tensor of large Lawvere theories need not exist in general. This is a size issue — if arities of operations are unbounded, then the terms over a given set of variables need not form a set. E.g., the sum $L_1 + L_2$ of almost any unranked large Lawvere theory L_1 and the theory L_2 generated by a single unary operation and no equations fails to exist [25]. Generally, the tensor has a better chance to exist than the sum, since it introduces additional equations, and in fact existence of the sum implies existence of the tensor [12]. Nevertheless, the tensor of two large Lawvere theories may fail to exist even when one of the component theories is ranked (of course, it does exist in case *both* components are ranked); e.g. there are theories whose tensor with the list theory fails to exist [18]. We proceed to show that the tensor exists whenever one of the component theories is uniform in the sense defined presently.

Definition 6 (Uniformity). Let L be a large Lawvere theory. The *constants* of L are the elements of $c_L := L(0, 1)$. For every set n we denote by $c_L^n : n \rightarrow n + c_L$ the morphism $[\text{id}] \times \prod_{f \in c_L} f$. We say that L is *uniform* if for every L -morphism $f : n \rightarrow m$ there exists a *generic morphism*, i.e. a morphism $\hat{f} : k \rightarrow 1$ for some set k such that there exists a set-function $u : k \times m \rightarrow n + c_L$ with $f = (\hat{f} \otimes m)[u]c_L^n$.

In other words, a theory is uniform if all terms over a given set n of variables can be obtained from a single generic term \hat{f} , possibly having more variables, by substituting for the variables of \hat{f} either variables from n or constants. The relevance to existence of tensors is clear: if a theory L_2 is uniform, then the tensor law of a putative tensor $L_1 \otimes L_2$ can always be made to apply to a term that has, say, a top layer of operations from L_1 whose arguments have a top layer from L_2 .

Remark 7. It is easy to see that in Definition 6, k can be bounded by $(n + c_L)^m$.

Example 8. 1) *The theory $L_{\mathcal{P}^*}$ of non-blocking unbounded nondeterminism is uniform:* Recall that a morphism $f : n \rightarrow m$ in $L_{\mathcal{P}^*}$ is a family of m nonempty subsets of n . As a generic morphism \hat{f} for f , we can thus take the full set n , seen as a morphism $n \rightarrow 1$, from which any other subset of n can be obtained by identifying some of the variables.

2) *The theory $L_{\mathcal{P}}$ of unbounded nondeterminism is uniform:* The argument is analogous as for $L_{\mathcal{P}^*}$, except that we now need to use also the constant \emptyset in substitutions in order to obtain the empty set as a substitution instance of the generic morphism \hat{f} .

3) *The theory $L_{\mathcal{P}_\omega}$ of finite nondeterminism fails to be uniform:* if $\sup_{i \in m} |A_i| = \infty$ for an infinite family $(A_i)_{i \in m}$ of finite subsets of an infinite set n , then there is no single finite set from which all sets A_i can be obtained by substituting \emptyset or variables from n .

4) *The theory $L_{\mathcal{P}_{\omega_1}}$ of countable nondeterminism is uniform:* any infinite countable subset of n will serve as a generic morphism \hat{f} for any morphism $f : n \rightarrow m$, i.e. any family of at m most countable subsets of n .

5) *The theory L_{mult} of unbounded weighted nondeterminism is uniform:* Recall from Example 2.2 that a morphism $f : n \rightarrow m$ in L_{mult} is a family of m multisets over n (i.e. maps $n \rightarrow \mathbb{N} \cup \{\infty\}$). As a generic morphism \hat{f} , we can take the multiset over $\mathbb{N} \times n$ that contains every element with multiplicity 1.

Moreover, uniformity also subsumes continuations, a fact that we state and prove separately:

Lemma 9. *For every R , the continuation theory L_{cont}^R (Example 2.3) is uniform.*

Proof: W.l.o.g. $|R| \geq 2$. We identify the set of constants of L_{cont}^R with R . Let $f : n \rightarrow m$ in L_{cont}^R ; recall that L_{cont}^R is the dual of the Kleisli category of the continuations monad, i.e. f is a map $m \rightarrow ((n \rightarrow R) \rightarrow R)$. Pick J such that $|m| \leq |R^J|$; we can assume w.l.o.g. that $m = R^J$, as we can just pad out $f : n \rightarrow m$, thought of as a family of m morphisms $n \rightarrow 1$, with sufficiently many copies of one of these morphisms. The required generic morphism for f is $\hat{f} : n + J \rightarrow 1$, defined by

$$\hat{f}(c) = f(\lambda j. c(\text{inr } j))(\lambda a. c(\text{inl } a))$$

for $c : n + J \rightarrow R$: Let $u : (n + J) \times m \rightarrow n + R$,

$$u(x, i) = \text{case } x \text{ of inl } y \mapsto \text{inl } y; \text{ inr } j \mapsto \text{inr } i(j).$$

Then for $i \in m = R^J$ and $k : n \rightarrow R$,

$$\begin{aligned} ((\hat{f} \otimes m)[u]c_L^n)(i)(k) &= \hat{f}(\lambda x. \text{case } u(x, i) \text{ of inl } y \mapsto k(y); \text{ inr } r \mapsto r) \\ &= \hat{f}(\lambda x. \text{case } x \text{ of inl } y \mapsto k(y); \text{ inr } j \mapsto i(j)) \\ &= f(i)(k). \end{aligned} \quad \square$$

The main existence result for tensors is as follows.

Theorem 10. *Let L_1, L_2 be large Lawvere theories, and let L_2 be uniform. Then the tensor product $L_1 \otimes L_2$ exists.*

Proof sketch: By explicit syntactic construction of the tensor product $L = L_1 \otimes L_2$. One constructs a precursor \mathbf{C} of the tensor whose morphisms $n \rightarrow m$ are equivalence

classes of paths $n \rightarrow m$; a single step $k \rightarrow l$ in a path is of the form $f * g$ where $f : p \rightarrow l$ in L_2 and $g : k \rightarrow p$ in L_1 . The equivalence is the congruence \sim on paths $\langle f_1 * g_1 \mid \dots \mid f_u * g_u \rangle$ generated by $\langle [\text{id}] * [\text{id}] \rangle \sim \langle \rangle$, $\langle f[e] * g \rangle \sim \langle f * [e]g \rangle$, and

$$\langle f * (n' \otimes g) \mid (f' \otimes m) * g' \rangle \sim \langle f(f' \otimes m') * (n \otimes g)g' \rangle.$$

Using uniformity of L_2 , one shows that every morphism of \mathbf{C} has a representative of the form

$$\langle f * g \mid c_{L_2}^n * [\text{id}] \rangle \quad (1)$$

(recall notation from Definition 6). One shows moreover that in (1), the domain of f can be taken to be $k = L_1(n + c_{L_2}, 1)$, so that \mathbf{C} is locally small. One defines a functor $I : \mathbf{Set}^{op} \rightarrow \mathbf{C}$ by $Ie = [e] * [\text{id}]$. It turns out that I maps products to *weak* products, i.e. factorizations through the product exist but need not be unique; this is amended by further quotienting. \square

Corollary 11. *For a large Lawvere theory L_1 , the tensor $L_1 \otimes L_2$ exists if L_2 is one of the following theories:*

- *unbounded or countable nondeterminism $L_{\mathcal{P}}, L_{\mathcal{P}_{\omega_1}}$;*
- *unbounded or countable non-blocking nondeterminism $L_{\mathcal{P}^*}, L_{\mathcal{P}_{\omega_1}^*}$;*
- *weighted nondeterminism L_{mult} ; or*
- *continuations L_{cont}^R .*

Of course, a corresponding result holds for monads. This result induces new monad transformers for nondeterminism, continuations, etc. The existence result for tensoring with continuations improves over previous results stating that the tensor of continuations with any *ranked* theory exists [12]. The results involving nondeterminism are, to our knowledge, entirely new. We refer to tensoring with any form of nondeterminism as a *powermonad* construction.

IV. COMPLETELY ADDITIVE MONADS AND THE FISCHER-LADNER ENCODING

Having shown that the tensor of any effect with nondeterminism always exists, we proceed to show that this amounts to a universal construction of an *additive* theory, i.e. a theory that includes nondeterministic choice operators which distribute over sequential composition. There are two versions of this phenomenon, with and without blocking (i.e. the empty set); for economy of presentation we concentrate on the case with blocking. We start out with a few notions concerned with blocking.

Definition 12 (Bounded theory). We call a large Lawvere theory L *bounded* if $|L(0, 1)| = 1$.

(The term *bounded* is motivated by the fact that \perp is the bottom of a natural *approximation ordering* introduced later.) Boundedness is connected to tensoring, as follows.

Definition 13. We denote by L_{\perp} the large Lawvere theory generated by a constant \perp and no equations.

Lemma 14. A large Lawvere theory L is bounded iff $L \otimes L_{\perp} \cong L$.

We denote the only constant of a bounded theory by $\perp_{0,1}$, and put $\perp_{n,m} = (\perp_{0,1} \otimes m)[\sharp] : n \rightarrow m$ for all n, m , where \sharp is the unique map $0 \rightarrow n$. In the sequel, we mostly write \perp in place of $\perp_{n,m}$. As usual, we have a corresponding notion of *bounded monad*.

Example 15 (Bounded effects). Besides L_{\perp} , basic examples of bounded theories include all forms of nondeterminism with blocking. Similarly, the list monad is bounded (\perp is the empty list). By Lemma 14, the state monad transformer, being defined by tensoring, preserves boundedness; e.g. the partial state monad $S \rightarrow (S \times _)\perp$ and the non-deterministic state monad $S \rightarrow \mathcal{P}(S \times _)$ are bounded.

Definition 16 (Additive theories). [26] A large Lawvere theory L is *finitely additive* if L is enriched over join semilattices, and *completely additive* if L is enriched over complete join semilattices (with \perp).

Again, corresponding notions for monads are implied. Joins serve to model nondeterministic choice. Recall that *enrichment* amounts to hom-sets carrying the structure in question, and composition preserving it in both arguments; i.e. composition distributes over choice and deadlock on both sides. The enrichment is a property rather than part of the data constituting a (completely or finitely) additive Lawvere theory, as by Lemma 24 proved later, the ordering is uniquely determined by the algebraic structure. The relation of additive theories to tensors is the following.

Lemma 17. For a large Lawvere theory L , the following are equivalent.

- i) L is completely additive.
- ii) $L \cong L \otimes L_{\mathcal{P}}$.
- iii) L is bounded and has a family of morphisms $U_n : n \rightarrow 1$, where n ranges over all sets, such that for any surjection $\sigma : m \rightarrow n$,

$$U_n = U_m[\sigma],$$

and for every L -morphism $f : m \rightarrow 1$,

$$f(U_n \otimes m) = U_n(n \otimes f).$$

The same equivalence holds for finitary additivity, tensoring with $L_{\mathcal{P}_{\omega}}$, and (iii) for finite n, m .

The operations U_n are n -fold joins, with 0-fold join U_0 necessarily being \perp . In other words, a completely additive theory L is one that has nondeterministic choice operators that commute over all operations of L as prescribed by the tensor law. From the above, it is immediate that

tensoring a large Lawvere theory L with $L_{\mathcal{P}}$ yields the free completely additive theory over L ,

i.e. the (overlarge) category of completely additive theories is reflexive in the category of large Lawvere theories.

Example 18. The generic example of a completely additive monad is \mathcal{P} , with joins being set unions. More generally, nondeterministic global state, $S \rightarrow \mathcal{P}(S \times _)$, is completely additive. A non-example is non-determinism with exceptions, $\mathcal{P}(_ + E)$, which has several constants and hence fails to be bounded.

Remark 19. Although completely additive theories are enriched, they can be treated as standard large Lawvere theories — as made explicit in Lemma 17, the completely additive structure is algebraic (although unranked), and hence respected by all product-preserving functors.

We proceed to formalize the example application from the introduction, i.e. to show that completely additive monads indeed allow for a generic Fischer-Ladner encoding of control structures. We base this formalization on the fact that every completely additive monad is a *Kleene monad* [26], i.e. supports Kleene iteration. Specifically, we can extend the computational metalanguage with operators \perp (deadlock), $+$ (binary choice, interpreted by binary joins), and a generic loop construct

$$\frac{\Gamma \triangleright p : TA \quad \Gamma, x : A \triangleright q : TA}{\Gamma \triangleright \text{init } x \leftarrow p \text{ in } q^* : TA}.$$

The latter is interpreted as the join of all finite iterations of q , prefixed with $x \leftarrow p$ and with the result x of the computation fed through the loop; i.e. $\text{init } x \leftarrow p \text{ in } q^*$ is the join of p and all programs $\text{do } x \leftarrow p; x \leftarrow q; \dots; x \leftarrow q; q$ where $x \leftarrow q$ appears $n \geq 0$ times. Moreover, we include in the signature the *test operator* $? : 2 \rightarrow T1$, which sends $\text{inr } *$ to $\text{ret } *$ and $\text{inl } *$ to \perp . Of course, $+$, \perp , and $?$ are supported already by finitely additive monads. From $+$, we can define an ordering \leq in the usual way via $p \leq q \iff p + q = q$. Issues in the axiomatization of a language with choice and iteration are studied in [26], [27]. Relevant axioms and rules for choice, deadlock, and iteration, including in particular two induction rules for iteration, are shown in Figure 1.

Recall that we have given a definition of the if-operator in terms of the case operator in the base language (Section II), with $2 = 1 + 1$ representing the Booleans. The proof of the following result appeals only to right distributivity of sequencing over choice and deadlock; we state it in less generality here only for the sake of brevity.

Proposition 20 (Generic Fischer-Ladner encoding).

- 1) Given a finitely additive monad T , for all $\Gamma \triangleright b : 2$, $\Gamma \triangleright p : TA$ and $\Gamma \triangleright q : TA$,

$$\text{if } b \text{ then } p \text{ else } q = \text{do } b?; p + \text{do } (\neg b)?; q.$$

(plus\perp)	$p + \perp = p$	(comm)	$p + q = q + p$
(idem)	$p + p = p$	(assoc)	$p + (q + r) = (p + q) + r$
(bind\perp_1)	$\text{do } x \leftarrow p; \perp = \perp$	(bind\perp_2)	$\text{do } x \leftarrow \perp; p = \perp$
(distr$_1$)	$\text{do } x \leftarrow p; (q + r) = \text{do } x \leftarrow p; q + \text{do } x \leftarrow p; r$		
(distr$_2$)	$\text{do } x \leftarrow (p + q); r = \text{do } x \leftarrow p; r + \text{do } x \leftarrow q; r$		
.....			
(unf$_1$)	$\text{init } x \leftarrow p \text{ in } q^* = p + \text{do } x \leftarrow (\text{init } x \leftarrow p \text{ in } q^*); q$		
(unf$_2$)	$\text{init } x \leftarrow p \text{ in } q^* = p + \text{init } x \leftarrow (\text{do } x \leftarrow p; q); \text{ in } q^*$		
(init)	$\text{init } x \leftarrow (\text{do } y \leftarrow p; q) \text{ in } r^* = \text{do } y \leftarrow p; \text{init } x \leftarrow q \text{ in } r^* \quad (y \notin FV(r))$		
(ind$_1$)	$\frac{\text{do } x \leftarrow p; q \leq p}{\text{init } x \leftarrow p \text{ in } q^* \leq p}$	(ind$_2$)	$\frac{\text{do } x \leftarrow q; r \leq r}{\text{do } x \leftarrow (\text{init } x \leftarrow p \text{ in } q^*); r \leq \text{do } x \leftarrow p; r}$

Figure 1. Axioms and rules for nondeterminism (top) and Kleene iteration (bottom)

2) Given a Kleene monad T , for every $\Gamma, x : A \triangleright b : 2$ and $\Gamma, x : A \triangleright p : TA$ the map sending $\Gamma, x : A \triangleright q$ to

$$\Gamma, x : A \triangleright \text{if } b \text{ then } \text{do } x \leftarrow p; q \text{ else } \text{ret } x$$

has a least fixed point given as the term

$$\text{do } x \leftarrow (\text{init } x \leftarrow \text{ret } x \text{ in } (\text{do } b?; p)^*); (-b)?; \text{ret } x.$$

The least fixed point in 2) is essentially a while loop, which could be denoted in the form $\text{while } b \text{ do } x \leftarrow p$. Here, the results of p are fed through the loop via the variable x ; note that the condition b itself does not read the state (being of type 2 instead of $T2$), but is updated in every iteration of the loop via its dependence on the loop variable x .

Remark 21. When absolutely free theories are used as a model of I/O, preservation of deadlock by sequencing from the left, as imposed by tensoring with \mathcal{P} , is hard to justify computationally. A satisfactory way of combining I/O with non-determinism and blocking will require fine-tuned mechanisms between sum and tensor yet to be developed.

V. CONSERVATIVITY

As indicated above, the algebraic complexity of the tensor implies that it is, in general, not at all clear that the component theories map faithfully into the tensor, i.e. that adding a new effect is *conservative*. In the following we investigate this issue for the powermonad obtained by tensoring with \mathcal{P} (nondeterminism with blocking); in the terminology of the previous section, this amounts to asking for which monads T it can be soundly assumed that they are completely additive, thus enabling, e.g., arguments using the Fischer-Ladner encoding.

Since $L_{\mathcal{P}}$ has a constant (\emptyset), an evident necessary condition for $L \rightarrow L \otimes L_{\mathcal{P}}$ to be faithful is that L can conservatively be made bounded, i.e. $L \rightarrow L \otimes L_{\perp}$ (Definition 13) must be faithful — this is equivalent to L being already

bounded if L has a constant, and a complex issue not in scope of the current investigation otherwise (one sufficient condition is that L is generated by equations having the same free variables on both sides). For the sake of readability, we thus restrict the further development to bounded large Lawvere theories.

However, constants are not the only problem: Even tensoring with nonempty powerset \mathcal{P}^* can be non-conservative, one counterexample being $(\mathcal{P}^*)^2 \otimes \mathcal{P}^* = \mathcal{P}^*$ where $(\mathcal{P}^*)^2 = \mathcal{P}^* \mathcal{P}^*$ is the double nonempty powerset monad (which may be thought of as generated by unbounded conjunction and disjunction operators and a distributive law). Collapse of $(\mathcal{P}^*)^2 \otimes \mathcal{P}^*$ to \mathcal{P}^* is due to a variant of the well-known Eckmann-Hilton argument [28].

We proceed to give an exact characterization of those theories L for which tensoring with nondeterminism $L_{\mathcal{P}}$ is order-theoretically conservative in a sense to be made precise presently. The main point here is that bounded large Lawvere theories carry a natural preordering:

Definition 22 (Approximation). Let L be a bounded large Lawvere theory. We compare elements of hom-sets $L(n, m)$ under the *approximation preorder* \sqsubseteq , which is the smallest preorder (strictly speaking: family of preorders on hom-sets) with \perp as a bottom element and closed under the rule

$$(\pi_{\sqsubseteq}) \quad \frac{\forall i. [\kappa_i]f \sqsubseteq [\kappa_i]g}{hf \sqsubseteq hg}$$

(equivalently, tupling and composition are monotone).

Roughly, $f \sqsubseteq g$ if f is obtained from g by repeatedly deleting subterms and applying the given equations. It is precisely the approximation preorder which provides the essential handle for characterizing conservativity. The approximation preorder does rely essentially on boundedness, i.e. on the fact that there is exactly one constant. To find a corresponding result for tensoring with non-empty

non-determinism (\mathcal{P}^*), i.e. to find a replacement for the approximation preorder in the absence of \perp , remains an open problem.

Example 23. 1) The approximation ordering on $L_{\mathcal{P}}$ and its variants is the subset relation. More generally, the approximation ordering coincides with the induced ordering in any additive theory, see Lemma 24 below.

2) In the list monad, $l \sqsubseteq k$ for lists l, k iff l can be obtained from k by deleting some of its entries.

3) The approximation ordering on the theory L_{mult} of weighted nondeterminism is multiset containment.

4) The approximation ordering on the partial state monad $S \rightarrow (S \times _)\perp$ is the extension ordering.

Lemma 24. *Let L be a finitely additive (hence bounded) large Lawvere theory. Then the approximation preorder \sqsubseteq on L coincides with the order \leq induced by the additive structure.*

Lemma 25. *Every morphism of bounded large Lawvere theories preserves the approximation preorder.*

The previous lemma applies in particular to the tensor map $L \rightarrow L \otimes L_{\mathcal{P}}$ for a bounded large Lawvere theory L . All this indicates that the relevant notion of conservativity should take into account the approximation preorder.

Definition 26. Let L be a bounded large Lawvere theory. We say that L admits unbounded nondeterminism if the tensor injection $\sigma_1 : L \rightarrow L \otimes L_{\mathcal{P}}$ is an order embedding, i.e. σ_1 is faithful and reflects the approximation ordering in the sense that $f \sqsubseteq g$ whenever $\sigma_1(f) \leq \sigma_1(g)$.

That is, L admits unbounded nondeterminism if tensoring L with $L_{\mathcal{P}}$ is order-theoretically conservative.

For the remainder of this section, let L be a bounded large Lawvere theory. In a first step, we apply two key simplifications to the description of $L \otimes L_{\mathcal{P}}$ given by the representation according to (1) (Section III): (1) and the subsequent reduction imply that we can represent a morphism $n \rightarrow m$ in $L \otimes L_{\mathcal{P}}$ as an m -tuple of sets of L -morphisms $n + \{\emptyset\} \rightarrow 1$ (which may be thought of as L -terms over $n + \{\emptyset\}$). We can, however

i) represent tuples of sets by sets of tuples using Cartesian products of sets, and

ii) get rid of occurrences of \emptyset in the bottom layer by replacing them with \perp , as $\perp = \emptyset$ in the tensor.

Based on these observations, we arrive at a construction of the tensor $L \otimes L_{\mathcal{P}}$ that can be proved correct independently of Theorem 10. To begin, we define a precursor of $L \otimes L_{\mathcal{P}}$, a category T_0 whose objects are sets and whose morphisms $n \rightarrow m$ are subsets of $L(n, m)$, with composition being complex multiplication $AB = \{fg \mid f \in A, g \in B\}$, and identities $\{\text{id}\}$. We have identity-on-objects functors $\sigma_1^0 : L \rightarrow T_0$ and $\sigma_2^0 : L_{\mathcal{P}} \rightarrow T_0$ defined by $\sigma_1^0 f = \{f\}$, and

by $\sigma_2^0(A_i) = \{[e] \mid e : m \rightarrow n, e(i) \in A_i \text{ for all } i\}$ for a morphism $(A_i) : n \rightarrow m$ in $L_{\mathcal{P}}$, i.e. a family of m subsets $A_i \subseteq n$. The category T_0 inherits a functor $I^0 : \mathbf{Set}^{op} \rightarrow T_0$ from L via σ_1^0 ; it is easy to see that under the axiom of choice (!), I^0 maps products to weak products.

We then define a relation \approx on the sets $T_0(n, m)$ (strictly speaking: a family of relations on hom-sets) inductively as the smallest equivalence that contains all instances of the axiom scheme

$$(\perp) \quad \{\perp_{n,m}\} \approx \emptyset \subseteq L(n, m)$$

and moreover forces uniqueness of tupling morphisms, i.e. is closed under the infinitary rule

$$(\pi) \quad \frac{\forall i. [\kappa_i]A \approx [\kappa_i]B}{CA \approx CB}$$

where L -morphisms (such as $[\kappa_i]$) are meant to convert to singletons when appropriate. We refer to \approx as *rectangular equivalence*. Implied properties of \approx are symmetry and congruence, the latter holding in particular for tupling and set union. We put $T = T_0/\approx$, and obtain functors $I : \mathbf{Set}^{op} \rightarrow T$, $\sigma_1 : L \rightarrow T$, $\sigma_2 : L_{\mathcal{P}} \rightarrow T$ by prolongation along $T_0 \rightarrow T$.

Theorem 27. *The category T of sets of L -morphisms modulo rectangular equivalence as constructed above is the tensor product $L \otimes L_{\mathcal{P}}$ of the bounded theory L with unbounded nondeterminism $L_{\mathcal{P}}$.*

Similar results hold for tensoring with $L_{\mathcal{P}^*}$ (in fact, the construction for $L_{\mathcal{P}^*}$ is slightly simpler) and for tensoring finitary theories with $L_{\mathcal{P}_\omega}$ or $L_{\mathcal{P}_\omega^*}$. Salient points in the proof are that the tensor law holds in T_0 up to rectangular equivalence, and moreover that the general tensor law justifies pointwise composition.

In L , we have morphisms $\Delta_i = \prod_j \delta_{ij} : n \rightarrow n$, where for $i, j \in n$, $\delta_{ij} : 1 \rightarrow 1$ equals $[\text{id}]$ if $i = j$ and \perp otherwise.

Lemma 28. *For $f : n \rightarrow m$, $g : m \rightarrow k$ in L , $fg \approx \{f\Delta_i g \mid i \in m\}$.*

Since the right hand side of the above equivalence is a join in the tensor $L \otimes L_{\mathcal{P}}$, order-theoretic conservativity will imply that it is a join already in L . We proceed to develop a characterization of the tensor in terms of order-theoretic closures from this observation.

Definition 29. We say that $A \subseteq L(n, m)$ is *closed* if A is downclosed and closed under the rule

$$(\Delta) \quad \frac{\forall i. g\Delta_i h \in A}{gh \in A}.$$

We denote the smallest closed set containing $A \subseteq L(n, m)$ by $\text{cl}(A)$. We write $\text{cl}(f)$ for $\text{cl}(\{f\})$.

The closure cl completely characterizes equality in the tensor:

Lemma 30. For $A, B \subseteq L(n, m)$, $A \approx B$ iff $\text{cl}(A) = \text{cl}(B)$.

To prove this core fact, we need a preliminary lemma:

Lemma 31. Let $A : n \rightarrow m$ in T_0 . Then for all $a : n \rightarrow m$ and all $b : m \rightarrow k$,

$$a \in \text{cl}(A) \implies ba \in \text{cl}(bA).$$

Proof: Show that the set $\{a \in \text{cl}(A) \mid ba \in \text{cl}(bA)\}$ contains A , is downward closed, and is closed under (Δ) . \square

Proof of Lemma 30: Only if: Show that the equivalence \simeq defined by $A \simeq B$ iff $\text{cl}(A) = \text{cl}(B)$ is closed under (π) and contains all instances of (\perp) . Here, left congruence can conveniently be split off from (π) as a separate rule, and closedness under left congruence is proved using Lemma 41.

If: It suffices to show that for $A : n \rightarrow m$ in T_0 , $A \approx \text{cl}(A)$. Since \approx is congruent w.r.t. set union, it suffices to show that $A \approx A \cup \{f\}$ for all $f \in \text{cl}(A)$, which will follow if we show that the set

$$\bar{A} := \{f : n \rightarrow m \mid A \approx A \cup \{f\}\}$$

(which clearly contains A) is downward closed and closed under Δ .

- \bar{A} is downward closed: define a preorder \leq by $f \leq g : \iff \{f, g\} \approx g$; then \leq is easily seen to be closed under π_{\sqsubseteq} , and hence contains \sqsubseteq . Now let $g \in \bar{A}$, $f \sqsubseteq g$. Then $f \leq g$ and therefore

$$A \cup \{f\} \approx A \cup \{g\} \cup \{f\} \approx A \cup \{g\} \approx A,$$

using congruence of \approx w.r.t. union.

- \bar{A} is closed under (Δ) : Let $f : n \rightarrow m$, $g : m \rightarrow k$, and let $f\Delta_i g \in \bar{A}$ for all $i \in m$. Then

$$A \cup \{fg\} \approx A \cup \{f\Delta_i g \mid i \in m\} \approx A,$$

using congruence w.r.t. union and Lemma 28. \square

Consequently, the tensor $L \otimes L_{\mathcal{P}}$ can be regarded as having closed subsets of $L(n, m)$ as morphisms $n \rightarrow m$. The following is, then, more or less immediate.

Theorem 32 (Order-theoretic conservativity). Let L be a bounded large Lawvere theory with approximation preorder \sqsubseteq as defined above, and let $\sigma_1 : L \rightarrow L \otimes L_{\mathcal{P}}$ be the tensor injection into the powermonad.

- 1) The following are equivalent:
 - (i) σ_1 reflects the approximation ordering.
 - (ii) For all $f : n \rightarrow m$, $g : m \rightarrow k$ in L , fg is a least upper bound of $\{f\Delta_i g \mid i \in m\}$.
 - (iii) For all $f : n \rightarrow m$ in L , $\text{cl}(f) = \{g \mid g \sqsubseteq f\}$.
- 2) If the equivalent conditions of 1) are satisfied, then σ_1 is monic iff \sqsubseteq is a partial order.

Summarizing the above, a bounded large Lawvere theory L admits unbounded nondeterminism iff the approximation

preorder on L is a partial order and for all $f : n \rightarrow m$, $g : m \rightarrow k$ in L ,

$$fg = \bigsqcup_{i \in m} f\Delta_i g.$$

Suprema of this form are preserved in the tensor.

Remark 33. There is a variant of Theorem 32 for tensoring finitary Lawvere theories with finite non-determinism, in which n, m, k are finite in (ii), (iii).

Remark 34 (Equational conservativity). One may wonder whether restricting to equational logic leads to weaker conditions for conservativity, which in the equational setting will be understood as faithfulness of the tensor map $\sigma_1 : L \rightarrow L \otimes L_{\mathcal{P}}$. However, the conditions for order-theoretic conservativity of Theorem 32 turn out to be necessary already for faithfulness of σ_1 under the mild additional assumption that L is simply ordered, i.e. given any upper bound h of $A \subseteq L(n, m)$, there exist f, g such that fg is an upper bound of A , $fg \sqsubseteq h$, and for every i there is $a \in A$ for which $f\Delta_i g \sqsubseteq a$. All example theories mentioned so far are simply ordered. For simply ordered theories, closed sets are closed under all existing suprema, similarly to Scott closed sets, and thus all existing suprema are preserved in the powermonad.

Example 35. All absolutely free theories L with at most one constant, such as input and output, map faithfully into $L \otimes L_{\perp}$, which can be shown to admit unbounded nondeterminism by Theorem 32 (see however Remark 21).

The partial state monad $S \rightarrow (S \times _)\perp$ admits unbounded nondeterminism, and Lemma 30 allows identifying the tensor as the nondeterministic state monad $S \rightarrow \mathcal{P}(S \times _)$ (this can also be obtained from the known description of tensors with the state monad [13]).

Every finitely additive finitary Lawvere theory admits unbounded nondeterminism. Hence, adding finite nondeterminism to a finitary theory is conservative iff adding unbounded nondeterminism is conservative.

Multisets do not admit nondeterminism: the upper bound of $\{a, \perp\}$ and $\{\perp, a\}$ is not $\{a, a\}$ but $\{a\}$. Similarly, lists do not admit unbounded nondeterminism, as $[a, b]$ is not a supremum of $[a, \perp] = [a]$ and $[\perp, b] = [b]$. In both cases, already faithfulness of the tensor map fails.

VI. CONCLUSION

We have proved the existence of tensors of large Lawvere theories for the case that one of the components is uniform. This implies in particular that one can always tensor with unbounded nondeterminism and with continuations, in the latter case improving a previous existence result [12]. We have then given a characterization of bounded theories that can be conservatively tensored with nondeterminism, which means precisely that one can assume such theories to be completely additive. Completely additive theories support a

calculus for Kleene iteration, in generalization of classical Kleene algebra, and, e.g., admit a generalized form of the classical Fischer-Ladner encoding [19].

Neither the present work nor [18] cover tensors with finite powerset, whose existence remains an open question. Although our results already have a quite order-theoretic flavour, an important issue for future research is whether similar results can be obtained in a domain-theoretic setting, using cpo-enriched Lawvere theories. Another direction for extending our results is to generalize them to enrichment over a topos, with a view to covering presheaf-based effects such as local state [29] or name creation [15].

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APPENDIX

(In the appendix, we use the term Lawvere theory to mean large Lawvere theory.)

A. Proof of Theorem 10.

The proof is by explicit syntactic construction of the tensor product $L = L_1 \otimes L_2$. To begin, we define a (not necessarily locally small) category \mathbf{C} on top of L_1, L_2 as follows. For $f \in \text{Hom}_{L_2}(k, m)$ and $g \in \text{Hom}_{L_1}(n, k)$ let $f * g$ be a synonym for the pair $\langle f, g \rangle$. We agree to omit the subscript at $*$ if it is clear from the context. We also agree that $*$ binds weaker than composition. Let us define objects of \mathbf{C} to be sets and morphisms from $\text{Hom}_{\mathbf{C}}(n, m)$ to be finite paths

$$\langle f_1 * g_1 \mid \dots \mid f_k * g_k \rangle$$

adhering to the typing constraints: n is the source of g_k , m is the target of f_1 , and for $i = 1, \dots, k-1$, the source of g_{i+1} is the target of f_i . We often omit brackets for one-element paths.

The identity morphisms of \mathbf{C} are the empty paths, and composition is concatenation of paths. Clearly, \mathbf{C} is a category. On every hom-set of \mathbf{C} we define an equivalence relation \sim as the equivalence generated by the clauses

$$\begin{aligned} \langle \dots \mid [\text{id}] * [\text{id}] \mid \dots \rangle &\sim \langle \dots \mid \dots \rangle, \\ \langle \dots \mid f[e] * g \mid \dots \rangle &\sim \langle \dots \mid f * [e]g \mid \dots \rangle \end{aligned}$$

and

$$\begin{aligned} \langle \dots \mid f * (n' \otimes g) \mid (f' \otimes m) * g' \mid \dots \rangle \\ \sim \langle \dots \mid f(f' \otimes m') * (n \otimes g)g' \mid \dots \rangle \end{aligned}$$

where $f' : n \rightarrow n'$ and $g : m \rightarrow m'$. By construction, \sim is a congruence on \mathbf{C} , so that we have a quotient category \mathbf{C}/\sim . Using the fact that L_2 is uniform, we show that every morphism of \mathbf{C}/\sim has a representative of the form

$$\langle f * g \mid c_{L_2}^n * [\text{id}] \rangle. \quad (2)$$

To that end, let us take any morphism f of \mathbf{C} . By attaching sufficiently many elements $[\text{id}] * [\text{id}]$ in the end of f we ensure that its length is at least 2 and its last element is $[\text{id}] * [\text{id}]$. Then we successively apply the following reduction sequence, whose net effect is length-decreasing, as long as possible:

$$\begin{aligned} \langle f_1 * g_1 \mid f_2 * g_2 \mid \dots \rangle \\ \sim \langle f_1 * g_1 \mid (\hat{f}_2 \otimes m_2)[u_{f_2}]c_{L_2}^{k_1} * g_2 \mid \dots \rangle \\ = \langle f_1 * g_1 \mid (\hat{f}_2 \otimes m_2)([u_{f_2}]c_{L_2}^{k_1} \otimes 1) \\ \quad * (k_1 \otimes [\text{id}])g_2 \mid \dots \rangle \\ \sim \langle f_1 * g_1 \mid (\hat{f}_2 \otimes m_2) * ((s \times m_2) \otimes [\text{id}]) \mid \\ \quad ([u_{f_2}]c_{L_2}^{k_1} \otimes 1) * g_2 \mid \dots \rangle \\ = \langle f_1 * g_1 \mid (\hat{f}_2 \otimes m_2) * [\text{id}] \mid [u_{f_2}]c_{L_2}^{k_1} * g_2 \mid \dots \rangle \end{aligned}$$

$$\begin{aligned} &\sim \langle f_1 * g_1 \mid (\hat{f}_2 \otimes m_2) * [u_{f_2}] \mid c_{L_2}^{k_1} * g_2 \mid \dots \rangle \\ &= \langle f_1 * (1 \otimes g_1) \mid (\hat{f}_2 \otimes m_2) * [u_{f_2}] \mid c_{L_2}^{k_1} * g_2 \mid \dots \rangle \\ &\sim \langle f_1(\hat{f}_2 \otimes k_1) * (s \otimes g_1)[u_{f_2}] \mid c_{L_2}^{k_1} * g_2 \mid \dots \rangle. \end{aligned}$$

Here, $f_i : k_i \rightarrow m_i$, $g_i : n_i \rightarrow k_i$ and $\hat{f}_2 : s \rightarrow 1$. At the last step of the reduction we obtain a pair of the form (2).

Let us define $I : \mathbf{Set}^{op} \rightarrow \mathbf{C}/\sim$ by putting $I(n) = n$ for every set n and $I(e) = [e] * [\text{id}]$ for every set-function $e : n \rightarrow m$. We would like to prove that I weakly preserves small products, i.e. the families $I(\kappa_i) : \sum_i n_i \rightarrow n_i$ define weak small products in \mathbf{C}/\sim . Let $f_i : m \rightarrow n_i$ be a family of morphisms in \mathbf{C}/\sim . First we consider a special case when every f_i is presentable by a one-element path, e.g. $f_i \sim g_i *_{k_i} h_i$. Since L_2 has all small products, there exists a morphism $h : m \rightarrow \sum_i k_i$ of L_2 such that for every i , $h_i = [\kappa_i]h$. Analogously, since L_1 has all small products, there exists a morphism $g : \sum_i k_i \rightarrow \sum_i n_i$ of L_1 such that for every i , $g_i[\kappa_i] = [\kappa_i]g$. The equality $f_i = I(\kappa_i)(g * h)$, characterizing weak products, now follows from the diagram

$$\begin{array}{ccccc} m & \xrightarrow{h_i} & k_i & \xrightarrow{g_i} & n_i \\ \downarrow h & & \uparrow [\kappa_i] & & \uparrow \\ \sum_i k_i & & & & \\ \downarrow g & & \uparrow [\kappa_i] & & \\ \sum_i n_i & & & & \end{array},$$

whose two cells are commuting in L_1 and in L_2 respectively. In general, the f_i might not be presentable by one-element paths, but as we have argued above, they must be presentable by paths of length 2. In particular, for every i , $f_i \sim g_i h_i$ where both $g_i : k_i \rightarrow n_i$ and $h_i : m \rightarrow k_i$ are one-element. As we have proved, there exists h such that for every i , $h_i = I(\kappa_i)h$. On the other hand, every $g_i I(\kappa_i)$ is easily seen to be equivalent to a one-element path and therefore, there exists g such that for every i , $g_i I(\kappa_i) = I(\kappa_i)g$. We have thus: $f_i \sim g_i h_i \sim g_i I(\kappa_i)h \sim I(\kappa_i)gh$ and we are done.

Let us prove that \mathbf{C}/\sim is locally small. Since every morphism of \mathbf{C} has the form (2), every hom-set of \mathbf{C}/\sim has at most as many equivalence classes as there are non-equivalent morphisms (2) in the corresponding hom-set of \mathbf{C} . Let us fix some pair $\langle f * g \mid c_{L_2}^n * [\text{id}] \rangle \in \text{Hom}_{\mathbf{C}}(n, m)$. Let $c = \text{Hom}_{L_2}(0, 1)$. By local smallness of L_2 , $s = \text{Hom}_{L_2}(n + c, 1)$ is a set, and thus $s \in \text{Ob}(L_1)$. For every $i \in k$, $g_i = [\kappa_i]g$ belongs to $\text{Hom}_{L_2}(n + c, 1)$ and we denote by $u : k \rightarrow s$ the induced index transformation. Let $h : n + c \rightarrow s$ be the tupling morphism for the whole family $\text{Hom}_{L_2}(n + c, 1)$. Then

$$\langle f * g \mid c_{L_2}^n * [\text{id}] \rangle \sim \langle f * g \mid [u]h \mid c_{L_2}^n * [\text{id}] \rangle$$

$$\sim (\llbracket f[u] \star_s h \mid c_{L_2}^n \star [\text{id}] \rrbracket).$$

We have thus shown that every morphism of $\text{Hom}_{\mathbf{C}/\sim}(n, m)$ has a representative in the set $\text{Hom}_{L_1}(m, s) \times \text{Hom}_{L_2}(n, s)$ and hence $\text{Hom}_{\mathbf{C}/\sim}(n, m)$ is also a set.

Let \approx be the smallest congruence on \mathbf{C} , containing \sim and closed under the rule:

$$\forall i. [\kappa_i]f \approx [\kappa_i]g \implies f \approx g. \quad (3)$$

We then have a canonical functor $\mathbf{C}/\sim \rightarrow \mathbf{C}/\approx$, which equips \mathbf{C}/\approx with all small weak products, which due to (3) are in fact products. By postcomposing $I : \mathbf{Set}^{op} \rightarrow \mathbf{C}/\sim$ with the canonical projection, we obtain a product preserving functor $I : \mathbf{Set}^{op} \rightarrow \mathbf{C}/\approx$, so that $L = \mathbf{C}/\approx$ is a Lawvere theory. We will be done once we show that $L = L_1 \otimes L_2$. We define functors $\sigma_i : L_i \rightarrow L$ by

$$\sigma_1(f) = f \star [\text{id}], \quad \sigma_2(g) = [\text{id}] \star g$$

(omitting equivalence class formation from the notation). The following calculation ensures commutativity of σ_1 and σ_2 :

$$\begin{aligned} (\sigma_2(g) \otimes n')(m \otimes \sigma_1(f)) &= (\llbracket [\text{id}] \star g \rrbracket \otimes n')(m \otimes (\llbracket f \star [\text{id}] \rrbracket)) \\ &= (\llbracket [\text{id}] \star g \otimes n' \rrbracket)(\llbracket m \otimes f \star [\text{id}] \rrbracket) \\ &= [\text{id}](m' \otimes f) \star (g \otimes n)[\text{id}] \\ &= (m' \otimes f) \star (g \otimes n) \\ &= (\llbracket m' \otimes f \star [\text{id}] \rrbracket)(\llbracket [\text{id}] \star g \otimes n \rrbracket) \\ &= (m' \otimes (\llbracket f \star [\text{id}] \rrbracket))(\llbracket [\text{id}] \star g \otimes n \rrbracket) \\ &= (m' \otimes \sigma_1(f))(\sigma_2(g) \otimes n). \end{aligned}$$

Finally, let L' be another Lawvere theory equipped with a pair of commuting morphisms $\alpha_i : L_i \rightarrow L'$. We define a morphism of categories $\alpha : \mathbf{C} \rightarrow L'$ to be identity on objects and by the equations $\alpha(\llbracket \cdot \rrbracket) = \text{id}$ and

$$\begin{aligned} \alpha(\llbracket f_1 \star g_1 \mid \dots \mid f_k \star g_k \rrbracket) &= \\ &= \alpha_1(f_1)\alpha_2(g_1) \dots \alpha_1(f_k)\alpha_2(g_k) \end{aligned}$$

on morphisms. It is straightforward to verify by definition that $f \approx g$ implies $\alpha(f) = \alpha(g)$. Therefore, by the characteristic property of the quotient category, α lifts to a morphism of Lawvere theories $\alpha : L \rightarrow L'$. It is again easy to verify that for $i = 1, 2$, $\alpha_i = \alpha\sigma_i$. Uniqueness of α is clear. Therefore L is indeed a tensor product of L_1 and L_2 and we are done. \square

B. Proof of Proposition 20

1) Let for every $\Gamma \triangleright t : TA \times TA$, $h(t) = \text{fst}(t) + \text{snd}(t)$. Then

$$\begin{aligned} \text{if } b \text{ then } p \text{ else } q &= \text{case } b \text{ of } \text{inl } \star \mapsto p; \text{inr } \star \mapsto q \\ &= \text{case } b \text{ of } \text{inl } \star \mapsto h\langle p, \perp \rangle; \text{inr } \star \mapsto h\langle \perp, q \rangle \end{aligned}$$

$$\begin{aligned} &= h(\text{case } b \text{ of } \text{inl } \star \mapsto \langle p, \perp \rangle; \text{inr } \star \mapsto \langle \perp, q \rangle) \\ &= h(\text{case } b \text{ of } \text{inl } \star \mapsto p; \text{inr } \star \mapsto \perp, \\ &\quad \text{case } b \text{ of } \text{inl } \star \mapsto \perp; \text{inr } \star \mapsto q) \\ &= \text{do}(\text{case } b \text{ of } \text{inl } \star \mapsto \text{ret } \star; \text{inr } \star \mapsto \perp); p + \\ &\quad \text{do}(\text{case } b \text{ of } \text{inl } \star \mapsto \perp; \text{inr } \star \mapsto \text{ret } \star); q \\ &= \text{do}(\text{case } b \text{ of } \text{inl } \star \mapsto \text{ret } \star; \text{inr } \star \mapsto \perp); p + \\ &\quad \text{do}(\text{case } \neg b \text{ of } \text{inl } \star \mapsto \text{ret } \star; \text{inr } \star \mapsto \perp); q \\ &= \text{do } b?; p + \text{do } (\neg b)?; q \end{aligned}$$

and we are done.

2) First, note that by Lemma 24, \sqsubseteq coincides with \leq . By part (i), we need to show that

$$\text{do } x \leftarrow (\text{init } x \leftarrow \text{ret } x \text{ in } (\text{do } b?; p)^*); (\neg b)?; \text{ret } x \quad (4)$$

is the least fixed point of

$$q \mapsto \text{do } b?; x \leftarrow p; q + \text{do } (\neg b)?; \text{ret } x. \quad (5)$$

First observe that (4) is a fixed point of (5):

$$\begin{aligned} \text{do } b?; x \leftarrow p; (\text{do } x \leftarrow (\text{init } x \leftarrow \text{ret } x \text{ in } (\text{do } b?; p)^*); \\ (\neg b)?; \text{ret } x) + \text{do } (\neg b)?; \text{ret } x \\ &= \text{do } x \leftarrow (\text{do } b?; x \leftarrow p; \\ &\quad \text{init } x \leftarrow \text{ret } x \text{ in } (\text{do } b?; p)^* + \text{ret } x); (\neg b)?; \text{ret } x \\ &= \text{do } x \leftarrow (\text{init } x \leftarrow (\text{do } b?; p) \text{ in } (\text{do } b?; p)^* \\ &\quad + \text{ret } x); (\neg b)?; \text{ret } x \\ &= \text{do } x \leftarrow (\text{init } x \leftarrow \text{ret } x \text{ in } (\text{do } b?; p)^*); (\neg b)?; \text{ret } x. \end{aligned}$$

In order to show that (4) is the least fixed point, suppose q is some other fixed point of (5). Then

$$\begin{aligned} \text{do } b?; x \leftarrow p; q &\leq q \\ \text{do } (\neg b)?; \text{ret } x &\leq q \end{aligned}$$

From the former inequality, by **(ind₂)**:

$$\text{do } x \leftarrow (\text{init } x \leftarrow \text{ret } x \text{ in } (\text{do } b?; p)^*); q \leq q$$

from which we conclude by the latter inequality,

$$\text{do } x \leftarrow (\text{init } x \leftarrow \text{ret } x \text{ in } (\text{do } b?; p)^*); (\neg b)?; \text{ret } x \leq q.$$

Therefore (4) is indeed the least fixed point of (5) and the proof is thus completed. \square

C. Proof of Lemma 24

To prove that \sqsubseteq is contained in \preceq , it suffices to show that \preceq has the closure properties defining \sqsubseteq . By definition, \preceq has \perp as bottom. To see that \preceq is closed under (π_{\sqsubseteq}) , let $f, g : n \rightarrow m$ in L such that $[\kappa_i]f \preceq [\kappa_j]g$ for all $i \in m$. By definition, this means that $[\kappa_i]f + [\kappa_j]g[\kappa_i](f + g) = [\kappa_j]g$ for all i , so that $f + g = g$, i.e. $f \preceq g$; since by definition, composition is monotone w.r.t. \preceq , it follows that $hf \preceq hg$ for $h : m \rightarrow k$.

To show that, conversely, \preceq is contained in \sqsubseteq , let $f \preceq g$. Then $f = f + \perp \sqsubseteq f + g = g$. \square

D. A Direct Construction of the Nonempty Powermonad

To pave the ground for the direct construction of the powermonad, i.e. the proof of Theorem 27, we describe the direct construction of the nonempty powermonad, i.e. tensoring with non-blocking unbounded nondeterminism.

We need a preliminary lemma to ease the proof of the tensor equation.

Lemma 36. *In the notation of Definition 4, the tensor equation reduces to the case $m_1 = m_2 = 1$.*

Proof: We prove the general case as follows: To check commutation of the requisite diagram for arbitrary m_1, m_2 , it suffices to check commutation for all postcompositions with the product projections $\pi_{ij} = [(i, j)] : m_1 \times m_2 \rightarrow 1$ for $i \in m_1, j \in m_2$ (where $(i, j) : 1 \rightarrow m_1 \times m_2$ denotes the obvious constant map). Note that $\pi_{ij} = \pi_i(m_1 \otimes \pi_j) = \pi_j(\pi_i \otimes m_2)$ where $\pi_i : m_1 \rightarrow 1$ and $\pi_j : m_2 \rightarrow 1$ are product projections. Therefore $\pi_{ij}(f_1 \otimes m_2) = \pi_i(m_1 \otimes \pi_j)(f_1 \otimes m_2) = \pi_i(f_1 \otimes \pi_j) = (\pi_i f_1 \otimes \pi_j)$ and $\pi_{ij}(m_1 \otimes f_2) = \pi_j(\pi_i \otimes m_2)(m_1 \otimes f_2) = \pi_j(\pi_i \otimes f_2) = \pi_i \otimes (\pi_j f_2)$. (Note here that for $f : n \rightarrow m$ and a map $e : k \rightarrow l$, $f \otimes [e] : n \times l \rightarrow m \times k$ is definable as the morphism into the k -fold product $m \times k$ whose postcomposition with the j -th projection $m \times k \rightarrow m$ ($j \in k$) is $f \pi_{e(j)}$, where $\pi_{e(j)}$ is the $e(j)$ -th product projection $n \times l \rightarrow n$.) Next note that $(\pi_i f_1 \otimes \pi_j)(n_1 \otimes f_2) = \pi_i f_1(n_1 \otimes \pi_j f_2)$ and $(\pi_i \otimes \pi_j f_2)(f_1 \otimes n_2) = \pi_j f_2(\pi_i f_1 \otimes n_2)$, so that we are done by commutation of

$$\begin{array}{ccc} n_1 \times n_2 & \xrightarrow{n_1 \otimes \pi_j f_2} & n_1 \\ \pi_i f_1 \otimes n_2 \downarrow & & \downarrow \pi_i f \\ n_2 & \xrightarrow{\pi_j f_2} & 1. \end{array}$$

□

Let L be a Lawvere theory. We give a construction of the tensor $T = L \otimes L_{\mathcal{P}^*}$. We begin by constructing a category T_0 with an identity-on-objects functor $I_0 : \mathbf{Set} \rightarrow T_0$, with the same notation as for Lawvere theories, with the following properties:

- I_0 maps products to weak products;
- T_0 has functors $F^0 : L \rightarrow T_0$, $F_{\mathcal{P}^*}^0 : L_{\mathcal{P}^*} \rightarrow T_0$ that commute with the respective functors from \mathbf{Set}

That is, T_0 will fail to be the tensor $L \otimes L_{\mathcal{P}^*}$ on two counts: tupling morphisms need not be unique in T_0 , and the two functors from the component theories into T_0 need not commute. One of the surprises in the construction is that repairing the first defect will remedy also the second one.

Morphisms $n \rightarrow m$ in T_0 are just nonempty sets of L -morphisms. Composition is defined by $AB = \{ab \mid a \in A, b \in B\}$; identities are singleton sets $\{\text{id}_n\}$. The functor $F^0 : L \rightarrow T_0$ maps a morphism f to the singleton $\{f\}$. We then define I_0 as the composite $\mathbf{Set} \rightarrow L \rightarrow T_0$. The

tupling $\langle A_i \rangle : n \rightarrow \sum m_i$ of k T_0 -morphisms $A_i : n \rightarrow m_i$ is defined as

$$\langle A_i \rangle = \{\langle f_i \rangle \mid f_i \in A_i \text{ for all } i\},$$

where $\langle f_i \rangle$ denotes tupling in L . We regard morphisms $n \rightarrow m$ in $L_{\mathcal{P}^*}$ as m -tuples (A_i) of nonempty subsets of n . Then the functor $F_{\mathcal{P}^*}^0 : L_{\mathcal{P}^*} \rightarrow T_0$ maps (A_i) to

$$F_{\mathcal{P}^*}^0((A_i)) = \{[e] \mid e : m \rightarrow n, e(i) \in A_i \text{ for all } i\}.$$

In the special case $m = 1$, in which case a morphism $n \rightarrow m$ is just a single subset $A \subseteq n$, note that $F_{\mathcal{P}^*}^0(A) = \{[\kappa_i] \mid i \in A\}$.

This completes the definition of T_0 . We need to check a few properties:

- *The composite $\mathbf{Set} \rightarrow L_{\mathcal{P}^*} \rightarrow T_0$ coincides with I_0 .* To see this, let $e : m \rightarrow n$ be a map. In $L_{\mathcal{P}^*}$, $[e] : n \rightarrow m$ is the m -tuple $(\{e(i)\})_{i \in m}$. Under $F_{\mathcal{P}^*}^0$, this becomes the set

$$\{[\bar{e}] \mid \bar{e} : m \rightarrow n, \bar{e}(i) \in \{e(i)\} \text{ for all } i\} = \{[e]\}.$$

- *The tupling morphisms project back to their components:* For $A_i : n \rightarrow m_i$ in T_0 , we have

$$\begin{aligned} \pi_j \langle A_i \rangle &= \{\pi_j \langle f_i \rangle \mid (f_i) \in \prod A_i\} = \\ &= \{f_j \mid (f_i) \in \prod A_i\} = A_j, \end{aligned}$$

as the A_i are nonempty (note that this requires the axiom of choice).

We now proceed to repair the mentioned defects by quotienting T_0 by an appropriate equivalence relation. We define the relation \approx as the smallest reflexive transitive relation closed under the infinitary rule

$$(\pi) \frac{\forall i. [\kappa_i]A \approx [\kappa_i]B}{CA \approx CB}.$$

Lemma 37. *The relation \approx satisfies the following properties.*

- 1) \approx is symmetric.
- 2) \approx is a congruence w.r.t. composition.
- 3) \approx is a congruence w.r.t. tupling.

Proof:

- 1) Put $\approx^s = \approx \circ \approx^{-}$, where \cdot^{-} denotes the inverse relation. Clearly, \approx^s is reflexive and transitive. Moreover, \approx^s is easily seen to be closed under (π) . Consequently, $\approx \subseteq \approx^s$, so that \approx is symmetric.

- 2) Let

$$A \approx^c B \iff \forall L, R. LAR \approx LBR.$$

Then \approx^c is clearly reflexive and transitive. Moreover, \approx^c is closed under (π) : if $[\kappa_i]A \approx^c [\kappa_i]B$ for all i , then in particular $[\kappa_i]AR \approx [\kappa_i]BR$ for all i, R and hence $LAR \approx LBR$ for all L, R . Therefore, \approx^c contains \approx , so that \approx is a congruence.

3) Let $A_i, B_i : n \rightarrow m_i$ and $A_i \approx B_i$ for $i \in k$. To prove $\langle A_i \rangle \approx \langle B_i \rangle$, we have to show $[\kappa_{(i,j)}] \langle A_i \rangle \approx [\kappa_{(i,j)}] \langle B_i \rangle$ for $i \in k, j \in m_i$ (i.e. $(i, j) \in \sum_{i \in k} m_i$). Now $\kappa_{(i,j)} = (\kappa_j \times \text{id}_k) \iota_i$, where $\iota_i : m_i \rightarrow \sum m_i$ is the coproduct embedding (and hence $[\iota_i]$ is a weak product projection). Thus, we have $[\kappa_{(i,j)}] \langle A_i \rangle = [\kappa_j] A_i \approx [\kappa_j] B_i = [\kappa_{(i,j)}] \langle B_i \rangle$, using the assumption $A_i \approx B_i$ and the fact that \approx is congruent w.r.t. composition.

A last observation that needs to be made is that the tensor law holds in T_0 modulo \approx : Let $A \subseteq m$, corresponding to the morphism $\bar{A} = \{[\kappa_i] \mid i \in A\} : m \rightarrow 1$ in T_0 , and let $f : n \rightarrow 1$ in L (identified with a singleton in T_0). We have to show that the diagram

$$\begin{array}{ccc} n \times m & \xrightarrow{n \otimes \bar{A}} & n \\ f \otimes m \downarrow & & \downarrow f \\ m & \xrightarrow{\bar{A}} & 1 \end{array} \quad (6)$$

commutes. Now we have

$$\begin{aligned} n \otimes \bar{A} &= \langle \bar{A}[\kappa_i \times \text{id}_m] \rangle_{i \in n} \\ &= \langle \{[\kappa_i \times \text{id}_m] \kappa_j \mid j \in A\} \rangle_{i \in n} \\ &= \langle \{[\kappa_{(i,j)}] \mid j \in A\} \rangle_{i \in n} \\ &= \langle \{[\kappa_{(i,j_i)}] \}_{i \in n} \mid (j_i) \in A^n \rangle \end{aligned}$$

and hence

$$f(n \otimes \bar{A}) = \langle f[\kappa_{(i,j_i)}] \rangle_{i \in n} \mid (j_i) \in A^n \rangle.$$

On the other hand, we have

$$\begin{aligned} \bar{A}(f \otimes m) &= \bar{A} \langle f[\text{id}_n \times \kappa_j] \rangle_{j \in m} \\ &= \{[\kappa_j] \langle f[\text{id}_n \times \kappa_j] \rangle_{j \in m} \mid j \in A\} \\ &= \langle f[\text{id}_n \times \kappa_j] \mid j \in A \rangle \end{aligned}$$

so that equivalence of the two sides follows from

$$\{[\text{id}_n \otimes \kappa_j] \mid j \in A\} \approx \langle \{[\kappa_{(i,j_i)}] \}_{i \in n} \mid (j_i) \in A^n \rangle. \quad (7)$$

To prove (7), we compare the projections along $[\kappa_i]$, $i \in n$, on both sides, and thus reduce the goal to the evident equality

$$\begin{aligned} \{[\kappa_i][\text{id}_n \times \kappa_j] \mid j \in A\} &= \{[\kappa_{(i,j)}] \mid j \in A\} \\ &= \{[\kappa_{(i,j_i)}] \mid (j_i) \in A^n\}. \end{aligned}$$

We have thus shown that $T = T_0 / \approx$ is a candidate for the tensor product of L and \mathcal{P}^* . It remains to prove the universal property. Thus, let S be a further candidate, i.e. a Lawvere theory with maps $G : L \rightarrow S, G_{\mathcal{P}^*} : L_{\mathcal{P}^*} \rightarrow S$ such that the tensor law is satisfied for G and $G_{\mathcal{P}^*}$. We define a functor $\bar{G} : T \rightarrow S$ as follows. In preparation, we note that every subset $A \subseteq n$ of some set n is a morphism $n \rightarrow 1$ in $L_{\mathcal{P}^*}$, whose image under $G_{\mathcal{P}^*}$ we denote by $(A \subseteq n) : n \rightarrow 1$.

In particular, for a morphism $A : n \rightarrow m$ in T_0 we have $\hat{A} := (A \subseteq L(n, m)) : L(n, m) \rightarrow 1$ in S . Moreover, we have for each set n a morphism

$$s_n = G(\langle f \rangle_{f \in L(n)}) : n \rightarrow L(n)$$

in S , where we denote by $L(n)$ the action of the monad induced by L , i.e. simply $L(n) = L(n, 1)$. We then define a functor $\bar{G}_0 : T_0 \rightarrow S$ by putting, for $A : n \rightarrow 1$ in T_0 ,

$$\bar{G}_0(A) = \hat{A} s_n.$$

In general, we then put

$$\bar{G}_0(A) = \langle G([\kappa_i]A) \rangle_{i \in m}$$

for $A : n \rightarrow m$ in T_0 (noting that this agrees with the previous definition in case $m = 1$).

To establish the requisite properties of \bar{G}_0 , we need the following lemma.

Lemma 38. *Let $A \subseteq n, B \subseteq m$, and let $e : n \rightarrow m$ such that $e[A] = B$. Then*

$$(A \subseteq n)[e] = (B \subseteq m)$$

in S .

Proof: Immediate from the corresponding equality in $L_{\mathcal{P}^*}$. \square

To begin, we now show that \bar{G}_0 preserves $[_]$, which will then also imply that \bar{G}_0 preserves identities. Thus, let $e : n \rightarrow m$ be a map; we have to show that $\bar{G}_0[e] = [e]$, which by applying product projections on both sides and by definition of \bar{G}_0 immediately reduces to the case $n = 1$, i.e. $e = \kappa_j$ for some $j \in m$. Now we have

$$\begin{aligned} \bar{G}_0[\kappa_j] &= \widehat{[\kappa_j]} s_m \\ &= (1 \subseteq 1)[\kappa_{[\kappa_j]}] s_m \\ &= \kappa_j, \end{aligned}$$

where the second step is by Lemma 38 (applied to $\kappa_{[\kappa_j]}[1] = \{[\kappa_j]\}$).

The crucial point in the proof is now to establish that \bar{G}_0 preserves composition. Again, this reduces immediately to the case where the codomain of the composite is 1. Thus, let $A : n \rightarrow 1$, and let $B : k \rightarrow n$ in T_0 ; put $B_j = [\kappa_j]B$ and $\nu_j = \lambda g. [\kappa_j]g : L(k, n) \rightarrow L(k)$ for $j \in n$. By Lemma 38, we then have $\hat{B}_j = \hat{B}[\nu_j]$. We start to transform $\bar{G}_0 A \bar{G}_0 B$:

$$\begin{aligned} \bar{G}_0 A \bar{G}_0 B &= \hat{A} s_n \langle \hat{B}_j s_k \rangle_{j \in n} \\ &= \hat{A} s_n \langle \hat{B}[\nu_j] s_k \rangle_{j \in n} \\ &= \hat{A} s_n (n \otimes \hat{B}) \langle [\nu_j] s_k \rangle_{j \in n} \\ &= \hat{A} (L(n) \otimes \hat{B}) (s_n \otimes L(k, n)) \langle [\nu_j] s_k \rangle_{j \in n}, \end{aligned}$$

using the tensor law in the last step. We proceed to analyse the right-hand subterm of the last term separately: we claim that

$$s_n \otimes L(k, n) \langle [\nu_j] s_k \rangle_{j \in n} = G \langle fg \rangle_{f \in L(n), g \in L(k, n)}. \quad (8)$$

We then note moreover that the right-hand side of (8) equals $[c]s_k$, where $c : L(n) \times L(k, n) \rightarrow L(k)$ is composition (this is proved by precomposing both sides with the projections $[\kappa_{(f,g)}]$: we have $[\kappa_{(f,g)}][c]s_k = [c\kappa_{(f,g)}]s_k = [\kappa_{fg}]s_k = G(fg)$.) We then conclude the argument by

$$\begin{aligned} \hat{A}(L(n) \otimes \hat{B})(s_n \otimes L(k, n)) \langle [\nu_j]s_k \rangle_{j \in n} \\ = \widehat{A \times B}[c]s_k = \widehat{AB}s_k, \end{aligned}$$

again using Lemma 38 in the last step.

It remains to prove our claim (8). We note that $n \times L(k, n)$ is the n -fold product of $L(k, n)$ in S , with projections $p_j = [\lambda g. (j, g)]$ for all $j \in n$, and at the same time the $L(k, n)$ -fold product of n , with projections $q_g = [\lambda j. (j, g)]$ for all $g \in L(k, n)$, and similarly $L(n) \times L(k, n)$ is the $L(k, n)$ -fold product of $L(n)$, with projections $\bar{q}_g = [\lambda f. (f, g)]$ for all $g \in L(k, n)$. We then prove (8) by precomposing both sides with $[\kappa_{(f,g)}]$. We have

$$\begin{aligned} [\kappa_{(f,g)}](s_n \otimes L(k, n)) \langle [\nu_j]s_k \rangle_{j \in n} \\ = [\kappa_f][\lambda f. (f, g)](s_n \otimes L(k, n)) \langle [\nu_j]s_k \rangle_{j \in n} \\ = [\kappa_f]\bar{q}_g(s_n \otimes L(k, n)) \langle [\nu_j]s_k \rangle_{j \in n} \\ = [\kappa_f]s_n q_g \langle [\nu_j]s_k \rangle_{j \in n} \\ = G(f)q_g \langle [\nu_j]s_k \rangle_{j \in n}. \end{aligned}$$

Thus we are done once we show that $q_g \langle [\nu_j]s_k \rangle_{j \in n} = G(g)$. To this end, we precompose both sides with $[\kappa_j]$ and calculate

$$\begin{aligned} [\kappa_j]q_g \langle [\nu_j]s_k \rangle_{j \in n} \\ = [\kappa_j][\lambda j. (j, g)] \langle [\nu_j]s_k \rangle_{j \in n} \\ = [\kappa_g][\lambda g. (j, g)] \langle [\nu_j]s_k \rangle_{j \in n} \\ = [\kappa_g]p_j \langle [\nu_j]s_k \rangle_{j \in n} \\ = [\kappa_g][\nu_j]s_k \\ = [\kappa_{[\kappa_j]g}]s_k = G([\kappa_j]g) = [\kappa_j]Gg. \end{aligned}$$

This concludes the proof that \bar{G}_0 preserves composition. It is then clear that \bar{G}_0 factors through T , as its kernel satisfies all properties featuring in the inductive definition of \approx (the kernel is, of course, reflexive and transitive, and it is closed under (π) because tupling is unique in S). Uniqueness of the arising factorizing morphism $\bar{G} : T \rightarrow S$ is clear, because every morphism $n \rightarrow m$ in T has the form $F_{\mathcal{P}^*}(A)F(B)$, where $B : n \rightarrow k$ in L and $A : k \rightarrow n$ in $L_{\mathcal{P}^*}$.

Summarizing the above, we have shown that

$$T_0 / \approx \text{ is the tensor product of } L \text{ and } \mathcal{P}^*.$$

One consequence of this is the following property:

Lemma 39. \approx is congruent w.r.t. union.

E. A Direct Construction of the Power Tensor

We proceed to give details for Theorem 27. The construction of tensor products with the full powerset theory $L_{\mathcal{P}}$ is similar to the one for the nonempty powerset, but more involved due to the fact that the full powerset theory has a constant, \emptyset . The general construction of tensoring a theory L with uniform theories tells us that for such a case, we have to expect a three-layered normal form that has operations of $L_{\mathcal{P}}$ on top, under this a layer of operations of L , and at the bottom a layer consisting not only of variables but possibly also of occurrences of \emptyset .

To simplify matters, we have assumed that the given theory L is bounded, with the unique constant denoted \perp . This will in particular allow us to replace occurrences of \emptyset in the bottom layer with \perp , thus effectively reverting to a two-layered structure.

Under the assumption that L is bounded, the tensor $L \otimes L_{\mathcal{P}}$ is constructed as follows. As in the case of the nonempty power tensor, we begin by constructing a preliminary category T_0 . Morphisms $n \rightarrow m$ in T_0 are (possibly empty) subsets of $L(n, m)$. Composition is pointwise, as previously; also, the definition of the embedding functors $L \rightarrow T_0$, $L_{\mathcal{P}} \rightarrow T_0$ from the component theories remains unchanged, similarly for the indexing functor $\mathbf{Set} \rightarrow T_0$. The crucial difference with the nonempty powerset theory is that we have to adapt the definition of tupling to work around the basic fact that Cartesian products of sets are empty if one of the sets is empty, i.e. Cartesian products do not directly provide a faithful representation of tuples of sets as sets of tuples. Here, we exploit the fact that \perp is available, and by the tensor law is equivalent to the empty set. We thus put, for $A : n \rightarrow m$ in T_0 ,

$$\tilde{A} = \begin{cases} \{\perp\} & \text{if } A = \emptyset \\ A & \text{otherwise} \end{cases}$$

and then define the tupling of a family of morphisms $A_i : n \rightarrow m$, $i \in k$, by

$$\langle A_i \rangle_{i \in k} = \{ \langle f_i \rangle_{i \in k} \mid \forall i \in k. f_i \in \tilde{A}_i \}.$$

By the calculation carried out for the nonempty case, we then have $\pi_j \langle A_i \rangle = \tilde{A}_j$ in T_0 , and subsequent quotienting will ensure that \tilde{A}_j becomes equal to A_j . Commutation of the tensor injections with the indexing functors is as in the nonempty case.

Next, we quotient T_0 by *rectangular equivalence*, i.e. the relation \approx defined inductively as the smallest reflexive and transitive relation closed under rule (π) and additionally satisfying the axiom

$$(\perp) \quad \{\perp_{n,m}\} \approx \emptyset$$

as well as the symmetric $\emptyset \approx \{\perp_{n,m}\}$ for all n, m . We have

Lemma 40. *The relation \approx satisfies the following properties.*

- 1) \approx is symmetric.
- 2) $A \approx \tilde{A}$ for all A .
- 3) \approx is a congruence w.r.t. composition.
- 4) \approx is a congruence w.r.t. tupling.

Proof:

- 1) As in the nonempty case.
- 2) Trivial.
- 3) Let

$$A \approx^c B \iff \forall L, R. LAR \approx LBR.$$

As in the nonempty case, \approx^c is easily seen to be reflexive, transitive, and closed under (π) . It remains to see that \approx^c contains all instances of (\perp) , i.e. (considering only one of the two symmetric cases of (\perp)) that for all $L, R, L\{\perp\}R \approx L\emptyset R = \emptyset$. But this follows from boundedness of L : for $l \in L, r \in R$, we have $l\perp r = \perp$, and therefore $L\{\perp\}R = \{\perp\} \approx \emptyset$. Therefore, \approx^c contains \approx , so that \approx is a congruence.

4) Let $A_i, B_i : n \rightarrow m_i$ and $A_i \approx B_i$ for $i \in k$. To prove $\langle A_i \rangle \approx \langle B_i \rangle$, we have to show $[\kappa_{(i,j)}]\langle A_i \rangle \approx [\kappa_{(i,j)}]\langle B_i \rangle$ for $i \in k, j \in m_i$. Now $\kappa_{(i,j)} = (\kappa_j \times \text{id}_k)\iota_i$, where $\iota_i : m_i \rightarrow \sum m_i$ is the coproduct embedding. Thus, we have $[\kappa_{(i,j)}]\langle A_i \rangle = [\kappa_j]\tilde{A}_i \approx [\kappa_j]A_i$ *poweq* $[\kappa_j]B_i \approx [\kappa_j]\tilde{B}_i = [\kappa_{(i,j)}]\langle B_i \rangle$, using the assumption $A_i \approx B_i$, congruence w.r.t. composition, and claim 2 of this lemma. \square

So far, we have established that $T := T_0/\approx$ is a large Lawvere theory that has theory morphisms $L \rightarrow T$ and $L_{\mathcal{P}} \rightarrow T$. To prove that T is a candidate for the tensor $L \otimes L_{\mathcal{P}}$, we need to show that the tensor law holds. The argument is mostly as in the nonempty case: we need only check those cases where the empty set can occur within a tupling operation; the only case in point is where $A = \emptyset$, in the notation of (6). This case, however, is taken care of by the fact that $\emptyset \approx \{\perp\}$ and by boundedness of L , which ensures that for $\{\perp\}$ in place of \bar{A} , both paths in (6) equal $\{\perp\}$.

It remains to prove the universal property. Given a further candidate S , i.e. a large Lawvere theory with morphisms $G : L \rightarrow S$ and $G_{\mathcal{P}} : L_{\mathcal{P}} \rightarrow S$ satisfying the tensor law, we define $\bar{G}_0 : T_0 \rightarrow S$ as before; the proof that \bar{G}_0 respects composition and $[_]$ is unchanged from the nonempty case. Again, it is clear that \bar{G}_0 factors through T because its kernel satisfies the inductive definition of \approx , including all instances of (\perp) as these are implied by validity of the tensor law in S . Uniqueness of the factorization is, again, clear.

F. Proof of Lemma 30

We need a preliminary lemma:

Lemma 41. *Let $A : n \rightarrow m$ in T_0 . Then for all $a : n \rightarrow m$ and all $b : m \rightarrow k$,*

$$a \in \text{cl}(A) \implies ba \in \text{cl}(bA). \quad (9)$$

Proof: It suffices to show that the set $\bar{A} = \{a \in \text{cl}(A) \mid ba \in \text{cl}(bA)\}$ contains A , is downward closed, and is closed under (Δ) . The first and second properties are clear; we check the third property. Thus, let $h : n \rightarrow k$ and $g : k \rightarrow m$ such that $g\Delta_j h \in \bar{A}$ for all $j \in k$. Then $bg\Delta_j h \in \text{cl}(bA)$ for all j , and hence $bgh \in \text{cl}(bA)$, so that $gh \in \bar{A}$ as required. \square

Proof of Lemma 30: In preparation, note that

$$\Delta_i = \langle \delta_{ij} \rangle_{j \in n} [\kappa_i]. \quad (10)$$

Only if: It suffices to show that the equivalence \simeq defined by $A \simeq B$ iff $\text{cl}(A) = \text{cl}(B)$ is closed under (π) and contains all instances of (\perp) . The latter holds by the definition of $\text{cl}(\emptyset)$. To check the former, we first show that \simeq is left congruent w.r.t. composition. Thus, let $A, B : n \rightarrow m$ and let $C : m \rightarrow k$ such that $\text{cl}(A) = \text{cl}(B)$. We have to show $\text{cl}(CA) = \text{cl}(CB)$. Since $\text{cl}(CB)$ is downward closed and closed under (Δ) , it suffices to prove $CA \subseteq \text{cl}(CB)$. Thus let $c \in C, a \in A$. Then $a \in \text{cl}(B)$ by assumption, and therefore $ca \in \text{cl}(cB) \subseteq \text{cl}(CB)$ by Lemma 41. It follows that $\text{cl}(CA) \subseteq \text{cl}(CB)$. The converse implication is shown symmetrically.

It remains to show that \simeq is closed under (π) . Thus, let $A, B : n \rightarrow m$ such that $\text{cl}([\kappa_i]A) = \text{cl}([\kappa_i]B)$ for all $i \in m$. We have to show $\text{cl}(A) = \text{cl}(B)$. By (10) and Lemma 41, we have $\Delta_i a \in \text{cl}((\Delta_i B))$ for all $i \in m, a \in A$. By downward closedness, $\text{cl}((\Delta_i B)) \subseteq \text{cl}(B)$, so that we obtain $a \in \text{cl}(B)$ by rule (Δ) . It follows that $\text{cl}(A) \subseteq \text{cl}(B)$; the reverse inclusion is shown symmetrically.

If: It suffices to show that for $A : n \rightarrow m$ in T_0 , $A \approx \text{cl}(A)$. Since \approx is congruent w.r.t. set union, it suffices to show that $A \approx A \cup \{f\}$ for all $f \in \text{cl}(A)$, which will follow if we show that the set

$$\bar{A} := \{f : n \rightarrow m \mid A \approx A \cup \{f\}\}$$

(which clearly contains A) is downward closed and closed under Δ .

- \bar{A} is downward closed: define a preorder \trianglelefteq by $f \trianglelefteq g : \iff \{f, g\} \approx g$; then \trianglelefteq is easily seen to be closed under π_{\square} , and hence contains \square . Now let $g \in \bar{A}, f \square g$. Then $f \trianglelefteq g$ and therefore

$$A \cup \{f\} \approx A \cup \{g\} \cup \{f\} \approx A \cup \{g\} \approx A,$$

using congruence of \approx w.r.t. union.

- \bar{A} is closed under (Δ) : Let $f : n \rightarrow m, g : m \rightarrow k$, and let $f\Delta_i g \in \bar{A}$ for all $i \in m$. Then

$$A \cup \{fg\} \approx A \cup \{f\Delta_i g \mid i \in m\} \approx A,$$

using congruence w.r.t. union and Lemma 28. \square

G. Proof of Lemma 28

By rule (π) , it suffices to prove $[\kappa_j]g \approx [\kappa_j]\{\Delta_i g \mid i \in m\}$ for all $j \in m$. But the right hand side of this equivalence equals $\{[\kappa_j]g\} \cup \{\perp\}$, which is equivalent to $[\kappa_j]g$ by (\perp) and congruence w.r.t. union. \square

H. Proof of the Order-Theoretic Conservativity Theorem 32

- 1) $a) \Rightarrow b)$: Immediate from Lemma 28.
 $b) \Rightarrow c)$: The inclusion $\text{cl}(f) \subseteq f\downarrow$ holds because under $b)$, $f\downarrow$ is closed under (Δ) . The reverse inclusion holds because $\text{cl}(f)$ is downclosed.
 $c) \Rightarrow a)$: Immediate by Lemma 30.
 2) Immediate from property $c)$. \square

I. Details for Remark 34

Let L be simply ordered. We prove that if $\sigma_1 : L \rightarrow L \otimes L_{\mathcal{P}}$ is faithful, then it reflects the ordering. According to Theorem 32, we have to show that for $f : n \rightarrow m$, $g : m \rightarrow k$ in L , $fg = \bigsqcup_{i \in m} f\Delta_i g$. Thus let h be an upper bound of $\{f\Delta_i g \mid i \in m\}$. Since L is simply ordered, there exist f', g' such that $f'g' \sqsubseteq h$, $f'g'$ is a minimal upper bound of $\{f\Delta_i g \mid i \in m\}$, and for every j there is i such that $f'\Delta_j g' \sqsubseteq f\Delta_i g \sqsubseteq fg$. By (Δ) , $f'g' \in \text{cl}(fg)$ and $fg \in \text{cl}(f'g')$. Hence $\text{cl}(f'g') = \text{cl}(fg)$. Faithfulness of σ_1 then implies that $fg = f'g' \sqsubseteq h$.

J. Details for Example 35

Absolutely free theories: For the sake of readability, we restrict to the case where L is generated by a finitary signature Σ (and, of course, no equations). Then $L \otimes L_{\perp}$ is generated by Σ and an additional constant \perp , and equations

$$f(\perp, \dots, \perp) = \perp$$

for every basic function symbol f in Σ . We can direct these equations from left to right and obtain a single-step rewrite relation \rightarrow which is clearly strongly normalizing and trivially locally confluent, hence confluent by Newman's lemma. We define a syntactic approximation \sqsubseteq_0 on $\Sigma \cup \{\perp\}$ -terms by $t \sqsubseteq_0 s$ iff there exist variables x_1, \dots, x_n such that $t = s[\perp/x_1, \dots, \perp/x_n]$; i.e. $t \sqsubseteq_0 s$ iff the term t is obtained from s by deleting some subterms. Then

Claim 42. $t \sqsubseteq s$ iff $NF(t) \sqsubseteq_0 s$

where $NF(t)$ denotes the normal form w.r.t. \rightarrow . 'If' is trivial since $t \sqsubseteq_0 s$ clearly implies $t \sqsubseteq s$.

To prove 'only if', first note that $t \sqsubseteq s$ iff s can be reached from t by a chain of terms $t = t_0, \dots, t_n = s$ such that for each $i = 0, \dots, n-1$, either $t_i \sqsubseteq_0 t_{i+1}$ or $t_i = t_{i+1}$ modulo the equations. We now show that whenever $t \sqsubseteq_0 s \leftrightarrow s'$, then there exists t' such that $t \rightarrow^* t' \sqsubseteq_0 s'$, where \leftrightarrow denotes the symmetric closure of \rightarrow , and \rightarrow^* the transitive reflexive closure. Since $s' \rightarrow s$ implies $s \sqsubseteq_0 s'$ and \sqsubseteq_0 is clearly a partial order, it suffices to consider the case $s \rightarrow s'$, that is, s' is obtained from s by deleting an occurrence r of a subterm of the form $f(\perp, \dots, \perp)$. If r is still present in t , we can delete it in t , obtaining t' such that $t \rightarrow t'$; then clearly $t' \sqsubseteq_0 s'$. If r is deleted in t , then already $t \sqsubseteq_0 s'$.

Now it follows that $t \sqsubseteq s$ iff there exists t' such that $t \rightarrow^* t' \sqsubseteq s$. We are done by noting that $NF(t) = NF(t') \sqsubseteq_0 t'$. This proves Claim 42.

We now set out to prove the equality $\bigsqcup_{i \in m} f\Delta_i g = fg$ for $g : n \rightarrow m$, $f : m \rightarrow k$. Translating this into the language of $\Sigma \cup \{\perp\}$ -terms, we have to show for a term t in variables x_1, \dots, x_m and terms s_i that $\bigsqcup t\sigma_i = t\sigma$, where the substitutions σ, σ_i are defined by $\sigma(x_i) = s_i$ and by $\sigma_i(x_i) = s_i, \sigma_i(x_j) = \perp$ for $i \neq j$, respectively. Thus, $t\sigma_i \sqsubseteq_0 t\sigma$ for all i , i.e. $t\sigma$ is an upper bound. Let h be a further upper bound of the $t\sigma_i$ w.r.t. \sqsubseteq . By Claim 42, $NF(t\sigma_i) \sqsubseteq_0 h$ for all i . We are done once we prove that $NF(t\sigma)$ is the least upper bound of the $NF(t\sigma_i)$ w.r.t. \sqsubseteq_0 . This just means that a subterm at position r in $t\sigma$ is deleted in $NF(t\sigma)$ iff it is deleted in all of the $t\sigma_i$. Here, 'only if' is clear (since the σ_i only introduce more occurrences of \perp). To prove 'if', we proceed as follows. If r lies within a substituted occurrence of x_i , i.e. within an occurrence of s_i , then the subterm at position r is the same in $t\sigma$ as in $t\sigma_i$, and we are done. Otherwise, the position r lies within t . Our goal then reduces to showing that for every term q in the variables $x_i, q\sigma_i = \perp$ (modulo the equations) for all i implies $q\sigma = \perp$. We prove this by induction on the term structure. The base case (where q is one of the variables x_i) is clear. If q is of the form $f(p_1, \dots, p_w)$, then $q\sigma_i = \perp$ implies (by confluence of \rightarrow) that $p_j\sigma_i = \perp$ for all i, j , so that $p_j\sigma = \perp$ for all j by induction, and hence $q\sigma = \perp$.

Finitely additive finitary Lawvere theories: Let L be a finitely additive finitary Lawvere theory; we prove that L admits unbounded nondeterminism. Recall that by Lemma 24, the approximation ordering on L coincides with the ordering induced by the additive structure. By Theorem 32, all that remains to be shown is that for all $f : n \rightarrow m, g : m \rightarrow k$ in L , $fg = \sum_{j \in m} f\Delta_j g$, where we can restrict to finite n, m, k because L is finitary. Since finite sums commute with composition, this reduces to showing that $\sum_{j \in m} \Delta_j = \text{id}_m$, which is straightforward by comparing projections.