A Modal Representation of Graded Medical Statements

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Abstract. Medical natural language statements uttered by physicians are usually *graded*, i.e., are associated with a degree of uncertainty about the validity of a medical assessment. This uncertainty is often expressed through specific *verbs*, *adverbs*, or *adjectives* in natural language. In this paper, we look into a *representation* of such graded statements by presenting a simple non-normal *modal logic* which comes with a set of modal operators, directly associated with the words indicating the uncertainty and interpreted through *confidence intervals* in the model theory. We complement the model theory by a set of RDFS-/OWL 2 RL-like entailment (*if-then*) rules, acting on the syntactic representation of modalized statements. Our interest in such a formalization is related to the use of OWL as the *de facto* standard in (medical) ontologies today and its weakness to represent and reason about assertional knowledge that is uncertain or that changes over time. The approach is not restricted to medical statements, but is applicable to other graded statements as well.

1 Introduction & Background

Medical natural language statements uttered by physicians or other health professionals and found in medical examination letters are usually *graded*, i.e., are associated with a degree of uncertainty about the validity of a medical assessment. This uncertainty is often expressed through specific verbs, adverbs, or adjectives in natural language (which we will call *gradation words*). E.g., *Dr. X suspects that Y suffers from Hepatitis* or *The patient probably has Hepatitis* or *(The diagnosis of) Hepatitis is confirmed*.
In this paper, we look into a representation of such graded statements by presenting a simple non-standard modal logic which comes with a small set of partially-ordered modal operators, directly associated with the words indicating the uncertainty and interpreted through confidence intervals in the model theory. The approach currently only addresses modalized propositional formulae in negation normal form which can be seen as a canonical representation of natural language sentences of the above form (a kind of a *controlled natural language*). Our interest in such a formalization is related to the use of OWL in our projects as the *de facto* standard for (medical) ontologies today and its weakness to represent and reason about assertional knowledge that is uncertain [15] or that

changes over time [12]. There are two principled ways to address such a restriction: *either* by sticking with the existing formalism (viz., OWL) and trying to find an encoding that still enables some useful forms of reasoning [15]; *or* by deviating from a defined standard in order to arrive at an easier, intuitive, and less error-prone representation [12].
Here, we follow the latter avenue, but employ and extend the standard entailment rules from [6] and [18] for positive binary relation instances in RDFS and OWL towards modalized n-ary relation instances, including negation. These entailment rules talk about, e.g., subsumption, class membership, or transitivity, and have been found useful in many applications. The proposed solution has been implemented in *HFC* [13], a forward chaining engine that builds Herbrand models which are compatible with the open-world view underlying OWL. The approach presented in this paper is clearly not restricted to medical statements, but is applicable to other graded statements as well (including *trust*), e.g., technical diagnosis (*The engine is probably overheated*) or more general in everyday conversation (*I'm pretty sure that X has signed a contract with Y*) which can be seen as the common case (contrary to true *universal* statements).

2 Graded Medical Statements: OWL vs. Modalized Representation

We note here that our initial modal operators were inspired by the *qualitative information parts* of diagnostic statements from [15] shown in Figure 1, but we might have chosen other operators, capturing the meaning of the gradation words used in the examples at the beginning of Section 1 (e.g., *probably*).

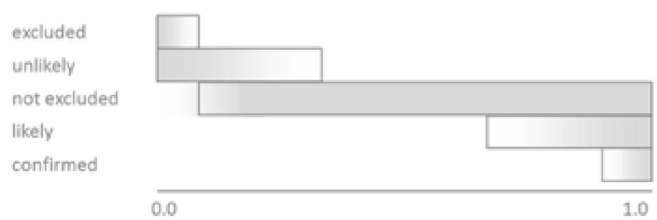


Fig. 1. Vague schematic mappings of the qualitative information parts *excluded* (E), *unlikely* (U), *not excluded* (N), *likely* (L), and *confirmed* (C) to *confidence intervals*, as used in this paper. Figure taken from [15].

These qualitative parts were used in statements about, e.g., liver inflammation with varying levels of detail. From this, we want to infer that, e.g., **if** *Hepatitis is confirmed* **then** *Hepatitis is likely* but **not** *Hepatitis is unlikely*. And **if** *Viral Hepatitis B is confirmed*, **then** both *Viral Hepatitis is confirmed* **and** *Hepatitis is confirmed* (generalization). Things "turn around" when we look at the adjectival modifiers *excluded* and *unlikely*: **if** *Hepatitis is excluded* **then** *Hepatitis is unlikely*, but **not** *Hepatitis is not excluded*. Furthermore, **if** *Hepatitis is excluded*,

then both *Viral Hepatitis is excluded* **and** *Viral Hepatitis B is excluded* (specialization). The set of *plausible* entailments for this kind of graded reasoning is depicted in Figure 2.

		Being said to have hepatitis (H) / viral hepatitis (vH) / viral hepatitis B (vHB) is…														
Precondition:		confirmed			likely			not excluded			unlikely			excluded		
Entailment:		H	vH	vHB	H	vH	vHB	H	vH	vHB	H	vH	vHB	H	vH	vHB
confirmed	H	x	x	x												
	vH		x	x												
	vHB			x												
likely	H	x	x	x	x	x	x									
	vH		x	x		x	x									
	vHB			x			x									
not excluded	H	x	x	x	x	x	x	x	x	x						
	vH		x	x		x	x		x	x						
	vHB			x			x			x						
unlikely	H										x			x		
	vH										x	x		x	x	
	vHB										x	x	x	x	x	x
excluded	H													x		
	vH													x	x	
	vHB													x	x	x

Fig. 2. Statements about liver inflammation with varying levels of detail: *Viral Hepatitis B* (vHB) implies *Viral Hepatitis* (vH) which implies *Hepatitis* (H). The matrix depicts entailments considered plausible, based on the inferences that follow from Figure 1. Hepatitis and its subclasses can be easily replaced by other medical situations/diseases. Figure taken from [15].

[15] consider five encodings (one outside the expressivity of OWL), from which only two were able to fully reproduce the inferences from Figure 2. Let us quickly look on approach 1, called *existential restriction*, before we informally present its modal counterpart (we will use abstract description logic syntax here [2]):

$\mathsf{HepatitisSituation} \equiv \mathsf{ClinicalSituation} \sqcap \exists \mathsf{hasCondition.Hepatitis}$

% *Hepatitis subclass hierarchy*

$\mathsf{ViralHepatitisB} \sqsubseteq \mathsf{ViralHepatitis} \sqsubseteq \mathsf{Hepatitis}$

% *vagueness via two subclass hierarchies*

$\mathsf{IsConfirmed} \sqsubseteq \mathsf{IsLikely} \sqsubseteq \mathsf{IsNotExcluded} \qquad \mathsf{IsExcluded} \sqsubseteq \mathsf{IsUnlikely}$

% *a diagnostic statement about Hepatitis*

$\mathsf{BeingSaidToHaveHepatitisIsConfirmed} \equiv \mathsf{DiagnosticStatement} \sqcap$
$\forall \mathsf{hasCertainty.IsConfirmed} \sqcap \exists \mathsf{isAboutSituation.HepatitisSituation}$

Standard OWL reasoning under this representation then ensures that, for instance,

$\mathsf{BeingSaidToHaveHepatitsIsConfirmed} \sqsubseteq \mathsf{BeingSaidToHaveHepatitisIsLikely}$

is the case, exactly one of the plausible inferences from Figure 2.

The encodings in [15] were quite cumbersome as the primary interest was to stay within the limits of the underlying calculus (OWL). Besides coming up with complex encodings, only minor forms of reasoning were possible, viz., subsumption reasoning. These disadvantages are a result of two conscious decisions:

OWL only provides unary and binary relations (concepts and roles) and comes up with a (mostly) fixed set of entailment/tableaux rules.
In our approach, however, the *qualitative information parts* from Figure 1 are first class citizens of the object language (the modal operators) and *diagnostic statements* from the Hepatitis use case are expressed through the binary property suffersForm between p (patients, people) and d (diseases, diagnoses). The plausible inferences are then simply a *byproduct* of the *instantiation* of the entailment rule schemas (G) from Section 5.1, and (S1) and (S0) from Section 5.2 for property suffersForm (the rule variables are universally quantified; $\top$ = *universal truth*; C = *confirmed*; L = *likely*), e.g.,

(S1) $\top\mathsf{ViralHepatitisB}(d) \wedge \mathsf{ViralHepatitisB} \sqsubseteq \mathsf{ViralHepatitis} \rightarrow \top\mathsf{ViralHepatitis}(d)$
(G) $C\mathsf{suffersFrom}(p, d) \rightarrow L\mathsf{suffersFrom}(p, d)$

Two things are worth to be mentioned here. *Firstly*, not only OWL-like properties (binary relations) can be graded, such as $C\mathsf{suffersFrom}(p, d)$ (= *it is confirmed that p suffers from d*), but also class membership (unary relations), e.g., $C\mathsf{ViralHepatitisB}(d)$ (= *it is confirmed that d is Viral Hepatitis B*). However, as the original OWL example above is unable to make use of any modals, we employ a special modal $\top$ here: $\top\mathsf{ViralHepatitisB}(d)$. *Secondly*, modal operators are only applied to assertional knowledge, involving individuals (the ABox in OWL)—neither axioms about classes (TBox) nor properties (RBox) are being affected by modals, as they are supposed to express universal truth.

3 Confidence of Statements and Confidence Intervals

We address the *confidence* of an asserted medical statement [15] through *graded* modalities applied to propositional formulae: E (*excluded*), U (*unlikely*), N (*not excluded*), L (*likely*), and C (*confirmed*). For various (technical) reasons, we add a *wildcard* modality ? (*unknown*), a complementary *failure* modality ! (*error*), plus two further modalities to syntactically state definite truth and falsity: $\top$ (*true*) and $\bot$ (*false*). Let $\triangle$ now denotes the set of all modalities:

$$\triangle = \{?, !, \top, \bot, E, U, N, L, C\}$$

A *measure function*

$$\mu : \triangle \mapsto [0, 1] \times [0, 1]$$

is a mapping which returns the associated *confidence interval* $[l, h]$ for a modality from $\triangle$ ($l \leq h$). We presuppose that

- $\mu(?) = [0, 1]$
- $\mu(!) = \emptyset$[3]
- $\mu(\top) = [1, 1]$
- $\mu(\bot) = [0, 0]$

In addition, we define two disjoint subsets of $\triangle$, called

- $\mathsf{1} = \{\top, C, L, N\}$
- $\mathsf{0} = \{\bot, E, U\}$

[3] Recall that an interval is a set of real numbers, together with a total ordering relation (e.g., $\leq$) over the elements, thus $\emptyset$ is a perfect, although degraded interval.

and again make a presupposition: the confidence intervals for modals from `1` *end* in 1, whereas the confidence intervals for `0` modals always *start* with 0. It is worth noting that we do *not* make use of μ in the syntax of the modal language (for which we employ the modalities from $\triangle$), but in the semantics when dealing with the satisfaction relation of the model theory (see Section 4).

We have talked about *confidence intervals* now several times without saying what we actually mean by this. Suppose that a physician says that it is *confirmed* (= C) that patient p suffers from disease d, given a set of recognized symptoms $S = \{s_1, \ldots, s_k\}$: $C\,\mathit{suffersFrom}(p, d)$.

Assuming that a different patient p' shows the same symptoms S (and only S, and perhaps further symptoms which are, however, *independent* from S), we would assume that the same doctor would diagnose $C\,\mathit{suffersFrom}(p', d)$.

Even an other, but similar trained physician is supposed to grade the two patients *similarly*. This similarity which originates from patients showing the same symptoms and from physicians being taught at the same medical school is addressed by confidence *intervals* and not through a *single* (posterior) probability, as there are still variations in diagnostic capacity and daily mental state of the physician. By using intervals (instead of single values), we can usually reach a consensus among people upon the *meaning* of gradation words, even though the low/high values of the confidence interval for, e.g., *confirmed* might depend on the context.

Being a bit more theoretic, we define a *confidence interval* as follows. Assume a *Bernoulli experiment* [11] that involves a large set of n patients P sharing the same symptoms S. W.r.t. our example, we would like to know whether $\mathit{suffersFrom}(p, d)$ or $\neg\mathit{suffersFrom}(p, d)$ is the case for every patient $p \in P$, sharing S. Given a Bernoulli trials sequence $\boldsymbol{X} = \langle x_1, \ldots, x_n \rangle$ with indicator random variables $x_i \in \{0, 1\}$ for a patient sequence $\langle p_1, \ldots, p_n \rangle$, we can *approximate* the *expected value* E for *suffersFrom* being *true*, given disease d and background symptoms S by the *arithmetic mean* A:

$$\mathrm{E}[\boldsymbol{X}] \approx \mathrm{A}[\boldsymbol{X}] = \frac{\sum_{i=1}^{n} x_i}{n}$$

Due to the *law of large numbers*, we expect that if the number of elements in a trials sequence goes to infinity, the arithmetic mean will coincide with the expected value:

$$\mathrm{E}[\boldsymbol{X}] = \lim_{n \to \infty} \frac{\sum_{i=1}^{n} x_i}{n}$$

Clearly, the arithmetic mean for each new *finite* trials sequence is different, but we can try to *locate* the expected value within an interval around the arithmetic mean:

$$\mathrm{E}[\boldsymbol{X}] \in [\mathrm{A}[\boldsymbol{X}] - \epsilon_1, \mathrm{A}[\boldsymbol{X}] + \epsilon_2]$$

For the moment, we assume $\epsilon_1 = \epsilon_2$, so that $\mathrm{A}[\boldsymbol{X}]$ is in the center of this interval which we will call from now on *confidence interval*.

Coming back to our example and assuming $\mu(C) = [0.9, 1]$, $C\,\mathit{suffersFrom}(p, d)$ can be read as being true in 95% of all cases *known* to the physician, involving patients p potentially having disease d and sharing the same prior symptoms (evidence) $s_1, \ldots, s_k$:

$$\frac{\sum_{p \in P} \mathrm{Prob}(\mathit{suffersFrom}(p,d) | s_1, \ldots, s_k)}{n} \approx 0.95$$

The variance of $\pm 5\%$ is related to varying diagnostic capabilities between (comparative) physicians, daily mental form, undiscovered important symptoms or examinations which have not been carried out (e.g., lab values), or perhaps even the physical stature of the patient which unconsciously affects the final diagnosis, etc, as elaborated above. Thus the individual modals from $\triangle$ express (via μ) different forms of the physician's confidence, depending on the set of already acquired symptoms as (potential) explanations for a specific disease.

4 Model Theory and Negation Normal Form

Let $\mathcal{C}$ denote the set of constants that serve as the arguments of a relation instance. In order to define basic n-ary propositional formulae (ground atoms, propositional letters), let $p(\boldsymbol{c})$ abbreviates $p(c_1, \ldots, c_n)$, for some $c_1, \ldots, c_n \in \mathcal{C}$, given $\mathit{length}(\boldsymbol{c}) = n$. In case the number of arguments do not matter, we sometimes simply write p, instead of, e.g., $p(c, d)$ or $p(\boldsymbol{c})$. As before, we assume $\triangle = \{?, !, \top, \bot, E, U, N, L, C\}$. We inductively define the set of *well-formed formulae* ϕ of our modal language as follows:

$$\phi ::= p(\boldsymbol{c}) \mid \neg\phi \mid \phi \wedge \phi' \mid \phi \vee \phi' \mid \triangle\phi$$

4.1 Simplification and Normal Form

We now syntactically *simplify* the set of well-formed formulae ϕ by restricting the uses of *negation* and *modalities* to the level of propositional letters p and call the resulting language Λ:

$$\pi ::= p(\boldsymbol{c}) \mid \neg p(\boldsymbol{c})$$

$$\phi ::= \pi \mid \triangle\pi \mid \phi \wedge \phi' \mid \phi \vee \phi'$$

To do so, we need the notion of a *complement* modal δ^{C} for every $\delta \in \triangle$, where

$$\mu(\delta^{\mathsf{C}}) := \mu(\delta)^{\mathsf{C}} = \mu(?) \setminus \mu(\delta) = [0,1] \setminus \mu(\delta)$$

I.e., $\mu(\delta^{\mathsf{C}})$ is defined as the complementary interval of $\mu(\delta)$ (within the bounds of $[0, 1]$, of course). For example, E and N (*excluded, not excluded*) or ? and ! (*unknown, error*) are already existing complementary modals. We also require *mirror* modals δ^{M} for every $\delta \in \triangle$ whose confidence interval $\mu(\delta^{\mathsf{M}})$ is derived by "mirroring" $\mu(\delta)$ to the opposite site of the confidence interval, either to the left or to the right:

$$\textbf{if } \mu(\delta) = [l, h] \textbf{ then } \mu(\delta^{\mathsf{M}}) := [1-h, 1-l]$$

For example, E and C (*excluded, confirmed*) or $\top$ and $\bot$ (*top, bottom*) are mirror modals. In order to transform ϕ into its *negation normal form*, we need to apply simplification rules a finite number of times (until rules are no longer applicable). We depict those rules by using the $\vdash$ relation, read as *formula* $\vdash$ *simplified formula*:

1. $?\phi \vdash \epsilon$ % *$?\phi$ is not informative at all, but its existence should alarm us*
2. $\neg\neg\phi \vdash \phi$
3. $\neg(\phi \wedge \phi') \vdash \neg\phi \vee \neg\phi'$
4. $\neg(\phi \vee \phi') \vdash \neg\phi \wedge \neg\phi'$
5. $\neg\triangle\phi \vdash \triangle^{\mathsf{C}}\phi$ (example: $\neg E\phi = N\phi$)
6. $\triangle\neg\phi \vdash \triangle^{\mathsf{M}}\phi$ (example: $E\neg\phi = C\phi$)

Clearly, the mirror modals δ^{M} are not necessary as long as we explicitly allow for negated statements, and thus case 6 can, in principle, be dropped.
What is the result of simplifying $\triangle(\phi \wedge \phi')$ and $\triangle(\phi \vee \phi')$? Let us start with the former case and consider as an example the statement about an engine that *a mechanical failure m and an electrical failure e is confirmed*: $C(m \wedge e)$. It seems plausible to simplify this expression to $Cm \wedge Ce$. Commonsense tells us furthermore that neither Em nor Ee is compatible with this description.
Now consider the "opposite" statement $E(m \wedge e)$ which must *not* be rewritten to $Em \wedge Ee$, as *either* Cm *or* Ce is well *compatible* with $E(m \wedge e)$. Instead, we rewrite this kind of "negated" statement as $Em \vee Ee$, and this works fine with either Cm or Ce.
In order to address the other modal operators, we generalize these plausible inferences by making a distinction between $\mathsf{0}$ and $\mathsf{1}$ modals (see Section 3):

7a. $\mathsf{0}(\phi \wedge \phi') \vdash \mathsf{0}\phi \vee \mathsf{0}\phi'$
7b. $\mathsf{1}(\phi \wedge \phi') \vdash \mathsf{1}\phi \wedge \mathsf{1}\phi'$

Now let us consider disjunction inside the scope of a modal operator. As we do allow for the full set of Boolean operators, we are allowed to deduce

8. $\triangle(\phi \vee \phi') \vdash \triangle(\neg(\neg(\phi \vee \phi'))) \vdash \triangle(\neg(\neg\phi \wedge \neg\phi')) \vdash \triangle^{\mathsf{M}}(\neg\phi \wedge \neg\phi')$

This is, again, a conjunction, so we apply schemas 7a and 7b, giving us

8a. $\mathsf{0}(\phi \vee \phi') \vdash \mathsf{0}^{\mathsf{M}}(\neg\phi \wedge \neg\phi') \vdash \mathsf{1}(\neg\phi \wedge \neg\phi') \vdash \mathsf{1}\neg\phi \wedge \mathsf{1}\neg\phi' \vdash \mathsf{1}^{\mathsf{M}}\phi \wedge \mathsf{1}^{\mathsf{M}}\phi' \vdash \mathsf{0}\phi \wedge \mathsf{0}\phi'$
8b. $\mathsf{1}(\phi \vee \phi') \vdash \mathsf{1}^{\mathsf{M}}(\neg\phi \wedge \neg\phi') \vdash \mathsf{0}(\neg\phi \wedge \neg\phi') \vdash \mathsf{0}\neg\phi \vee \mathsf{0}\neg\phi' \vdash \mathsf{0}^{\mathsf{M}}\phi \vee \mathsf{0}^{\mathsf{M}}\phi' \vdash \mathsf{1}\phi \vee \mathsf{1}\phi'$

Note how the modals from $\mathsf{0}$ in 7a and 8a act as a kind of *negation* to turn the logical operators into their counterparts, similar to de Morgan's law.

4.2 Model Theory

In the following, we extend the standard definition of modal (Kripke) frames and models [3] for the *graded* modal operators from $\triangle$ by employing the measure function μ and focussing on the minimal definition for ϕ in Λ. A *frame* $\mathcal{F}$ for the probabilistic modal language Λ is a pair

$$\mathcal{F} = \langle \mathcal{W}, \triangle \rangle$$

where $\mathcal{W}$ is a non-empty set of *worlds* (or *situations, states, points, vertices*) and $\triangle$ a family of binary relations over $\mathcal{W} \times \mathcal{W}$, called *accessibility relations*. Note that we have overloaded $\triangle$ (and each $\delta \in \triangle$) in that it refers to the modals used in the syntax of Λ, but also to depict the binary relations, connecting worlds.

A *model* $\mathcal{M}$ for the probabilistic modal language Λ is a triple

$$\mathcal{M} = \langle \mathcal{F}, \mathcal{V}, \mu \rangle$$

such that $\mathcal{F}$ is a *frame*, $\mathcal{V}$ a *valuation*, assigning each proposition ϕ a subset of $\mathcal{W}$, viz., the set of worlds in which ϕ holds, and μ a mapping, returning the confidence interval for a given modality from $\triangle$. Note that we only require a definition for μ in $\mathcal{M}$ (the model, but *not* in the frame), as $\mathcal{F}$ represent the relational structure without interpreting the edge labeling (the modal names) of the graph.
The *satisfaction relation* $\models$, given a model $\mathcal{M}$ and a specific world w is inductively defined over the set of well-formed formulae of Λ in *negation normal form* (remember $\pi ::= p(\boldsymbol{c}) \mid \neg p(\boldsymbol{c})$):

1. $\mathcal{M}, w \models p(\boldsymbol{c})$ **iff** $w \in \mathcal{V}(p(\boldsymbol{c}))$ **and** $w \notin \mathcal{V}(\neg p(\boldsymbol{c}))$
2. $\mathcal{M}, w \models \neg p(\boldsymbol{c})$ **iff** $w \in \mathcal{V}(\neg p(\boldsymbol{c}))$ **and** $w \notin \mathcal{V}(p(\boldsymbol{c}))$
3. $\mathcal{M}, w \models \phi \wedge \phi'$ **iff** $\mathcal{M}, w \models \phi$ **and** $\mathcal{M}, w \models \phi'$
4. $\mathcal{M}, w \models \phi \vee \phi'$ **iff** $\mathcal{M}, w \models \phi$ **or** $\mathcal{M}, w \models \phi'$
5. **for all** $\delta \in \triangle$: $\mathcal{M}, w \models \delta\pi$ **iff** $\frac{\#\{u \mid (w,u) \in \delta \textbf{ and } \mathcal{M}, u \models \pi\}}{\#\{u \mid (w,u) \in \delta' \textbf{ and } \delta' \in \triangle\}} \in \mu(\delta)$

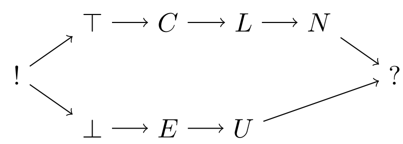
The last case of the satisfaction relation addresses the modals: for a world w, we look for the successor states u that are directly reachable via δ and in which π holds, and divide the number of such states by the number of all worlds that are directly reachable from w. This number between 0 and 1 must lie in the confidence interval $\mu(\delta)$ of δ in order to satisfy $\delta\pi$, given $\mathcal{M}, w$.
It is worth noting that the satisfaction relation above differs in its handling of $\mathcal{M}, w \models \neg p(\boldsymbol{c})$, as negation is *not* interpreted through the *absence* of $p(\boldsymbol{c})$ ($\mathcal{M}, w \not\models p(\boldsymbol{c})$), but through the *existence* of $\neg p(\boldsymbol{c})$. This treatment addresses the *open-world* nature in OWL and the evolvement of a (medical) domain over time.
We also note that the definition of the satisfaction relation for modalities (last clause) is related to the *possibility operators* $M_k\cdot$ ($= \Diamond^{\geq k}\cdot$; $k \in \mathbb{N}$) [4] and *counting modalities* $\cdot \geq n$ [1], used in modal logic characterizations of *description logics* with *cardinality* restrictions.

4.3 Well-Behaved Frames

As we will see later, it is handy to assume that the graded modals are arranged in a kind of hierarchy—the more we move "upwards" in the hierarchy, the more a statement in the scope of a modal becomes *uncertain*. In order to address this, we slightly extend the notion of a *frame* by a third component $\preceq \subseteq \triangle \times \triangle$, a partial order between modalities:

$$\mathcal{F} = \langle \mathcal{W}, \triangle, \preceq \rangle$$

Let us consider the following modal hierarchy that we build from the set $\triangle$ of already introduced modals:



This graphical representation is just a compact way to specify a set of 33 binary relation instances over $\triangle$, such as, e.g., $\top \preceq \top$, $\top \preceq N$, $C \preceq N$, $\bot \preceq ?$, or $! \preceq ?$. The above mentioned form of uncertainty is expressed by the measure function μ in that the associated confidence intervals become larger:

$$\textbf{if } \delta \preceq \delta' \textbf{ then } \mu(\delta) \subseteq \mu(\delta')$$

In order to arrive at a proper and intuitive model-theoretic semantics which mirrors intuitions such as **if** ϕ *is confirmed* $(C\phi)$ **then** ϕ *is likely* $(L\phi)$, we will focus here on *well-behaved* frames $\mathcal{F}$ which enforce the existence of edges in $\mathcal{W}$, given $\preceq$ and $\delta, \delta^{\uparrow} \in \triangle$:

$$\textbf{if } (w,u) \in \delta \textbf{ and } \delta \preceq \delta^{\uparrow} \textbf{ then } (w,u) \in \delta^{\uparrow}$$

However, by imposing this constraint, we also need to adapt the last case of the satisfiability relation:

5. **for all** $\delta \in \triangle$: $\mathcal{M}, w \models \delta\pi$ **iff** $\frac{\#\{u \mid (w,u) \in \delta^{\uparrow}, \delta \preceq \delta^{\uparrow}, \textbf{ and } \mathcal{M}, u \models \pi\}}{\#\{u \mid (w,u) \in \delta' \textbf{ and } \delta' \in \triangle\}} \in \mu(\delta)$

Not only are we scanning for edges (w,u) labeled with δ and for successor states u of w in which π holds in the denominator (original definition), but also take into account edges marked with more general modals $\delta^{\uparrow}$, s.t. $\delta^{\uparrow} \succeq \delta$. This mechanism implements a kind of *built-in model completion* that is not necessary in ordinary modal logics as they deal with only a *single* relation (viz., unlabeled arcs) that connects elements from $\mathcal{W}$ and the two modals $\Diamond$ and $\Box$ are defined in the usual dual way: $\Box\phi \equiv \neg\Diamond\neg\phi$.

5 Entailment Rules

This section addresses a restricted subset of entailment rules which will unveil new (or implicit) knowledge from graded medical statements. Recall that these kind of statements (in negation normal form) are a consequence of the application of simplification rules as depicted in Section 4.1. Thus, we assume a *preprocessing step* here that "massages" more complex statements that arise from a representation of graded (medical) statements in *natural language*. The entailments which we will present in a moment can either be *directly* implemented in a tuple-based reasoner, such as *HFC*, or in triple-based engines (e.g., Jena, OWLIM) which need to *reify* the medical statements in order to be compliant with the RDF triple model.

5.1 Modal Entailments

The entailments presented in this section deal with *plausible* inference centered around modals $\delta, \delta' \in \triangle$, some of them partly addressed in [15] in a pure OWL setting. We use the implication sign $\rightarrow$ to depict the entailment rules

$$lhs \rightarrow rhs$$

which act as *completion* (or *materialization*) rules the way as described in, e.g., [6] and [18], and used in today's semantic repositories. We sometimes even use the bi-conditional $\leftrightarrow$ to address that the LHS and the RHS are semantically equivalent, but will indicate the direction that should be used in a practical setting. As before, we define $\pi ::= p(\boldsymbol{c}) \mid \neg p(\boldsymbol{c})$.
We furthermore assume that for every modal $\delta \in \triangle$, a *complement* modal δ^{C} and a *mirror* modal δ^{M} exist (see Section 4.1).

Lift

$$(\mathsf{L}) \quad \pi \leftrightarrow \top\pi$$

This rule interprets propositional statements as special modal formulae. It might be dropped and can be seen as a pre-processing step. We have used it in the Hepatitis example above. Usage: left-to-right direction.

Generalize

$$(\mathsf{G}) \quad \delta\pi \wedge \delta \preceq \delta' \rightarrow \delta'\pi$$

This rule schema can be instantiated in various ways, using the modal hierarchy from Section 4.3; e.g., $\top\pi \rightarrow C\pi$, $C\pi \rightarrow L\pi$, or $E\pi \rightarrow U\pi$. It has been used in the Hepatitis example.

Complement

$$(\mathsf{C}) \quad \neg\delta\pi \leftrightarrow \delta^{\mathsf{C}}\pi$$

In principle, (C) is not needed in case the statement is already in negation normal form. This schema might be useful for natural language paraphrasing (explanation). Given $\triangle$, there are two possible instantiations, viz., $E\pi \leftrightarrow \neg N\pi$ and $N\pi \leftrightarrow \neg E\pi$ (note: $\mu(E) \cup \mu(N) = [0, 1]$).

Mirror

$$(\mathsf{M}) \quad \delta\neg\pi \leftrightarrow \delta^{\mathsf{M}}\pi$$

Again, (M) is in principle not needed as long as the modal proposition is in negation normal form, since we do allow for negated propositional statements $\neg p(\boldsymbol{c})$. This schema might be useful for natural language paraphrasing (explanation). For $\triangle$, there are six possible instantiations, viz., $E\pi \leftrightarrow C\neg\pi$, $C\pi \leftrightarrow E\neg\pi$, $L\pi \leftrightarrow U\neg\pi$, $U\pi \leftrightarrow L\neg\pi$, $\top\pi \leftrightarrow \bot\neg\pi$, and $\bot\pi \leftrightarrow \top\neg\pi$.

Uncertainty

$$(\mathsf{U}) \quad \delta\pi \wedge \neg\delta\pi \leftrightarrow \delta\pi \wedge \delta^{\mathsf{C}}\pi \leftrightarrow ?\pi$$

The *co-occurrence* of $\delta\pi$ and $\neg\delta\pi$ does *not* imply logical *inconsistency* (propositional case: $\pi \wedge \neg\pi$), but leads to complete *uncertainty* about the validity of π. Remember that $\mu(?) = \mu(\delta) \cup \mu(\delta^{\mathsf{C}}) = [0, 1]$ (usage: left-to-right direction):

$$\begin{array}{l} \quad\;\; 0 \qquad\qquad\qquad\quad 1 \\ \mu : |\!-\!\delta^{\mathsf{C}}\!-\!|\!-\!-\!\delta\!-\!-\!| \\ \qquad\;\; \pi \qquad\quad \pi \end{array}$$

Negation

$$\text{(N)} \quad \delta(\pi \wedge \neg\pi) \leftrightarrow \delta\pi \wedge \delta\neg\pi \leftrightarrow \delta\pi \wedge \delta^{\mathsf{M}}\pi \leftrightarrow \delta^{\mathsf{M}}\neg\pi \wedge \delta^{\mathsf{M}}\pi \leftrightarrow \delta^{\mathsf{M}}(\pi \wedge \neg\pi)$$

(N) shows that $\delta(\pi \wedge \neg\pi)$ can be formulated equivalently using the mirror modal:

$$\begin{array}{l} \quad 0 \qquad\qquad\qquad\qquad\quad 1 \\ \mu : |\text{—}\delta^{\mathsf{M}}\text{—}|\text{———}|\text{—}\,\delta\,\text{—}| \\ \quad \pi \wedge \neg\pi \qquad\quad \pi \wedge \neg\pi \end{array}$$

In general, (N) is *not* the modal counterpart of the *law of non-contradiction*, as $\pi \wedge \neg\pi$ is usually afflicted by vagueness, meaning that from $\delta(\pi \wedge \neg\pi)$, we can *not* infer that $\pi \wedge \neg\pi$ is the case for the concrete example in question (recall the intention behind the confidence intervals; see Section 3). There is one notable exception, involving the $\top$ and $\bot$ modals. This is formulated by the next entailment rule.

Error

$$\text{(E)} \quad \top(\pi \wedge \neg\pi) \leftrightarrow \bot(\pi \wedge \neg\pi) \rightarrow\ !(\pi \wedge \neg\pi)$$

(E) *is* the modal counterpart of the *law of non-contradiction* (recall: $\top = \bot^{\mathsf{M}}$ and $\bot = \top^{\mathsf{M}}$). For this reason and by definition, the *error* (or *failure*) modal ! from Section 3 comes into play here. The modal ! can serve as a hint to either stop a computation the first time it occurs or to continue reasoning, but to syntactically memorize the ground atoms (viz., π and $\neg\pi$) which have led to an inconsistency. Usage: left-to-right direction.

5.2 Subsumption Entailments

As before, we define two subsets of $\triangle$, called $\mathsf{1} = \{\top, C, L, N\}$ and $\mathsf{0} = \{\bot, E, U\}$, thus $\mathsf{1}$ and $\mathsf{0}$ effectively become

$$\mathsf{1} = \{\top, C, L, N, U^{\mathsf{C}}\} \qquad\qquad \mathsf{0} = \{\bot, U, E, C^{\mathsf{C}}, L^{\mathsf{C}}, N^{\mathsf{M}}\}$$

due to the use of complement modals δ^{C} and mirror modals δ^{M} for every base modal $\delta \in \triangle$ and by assuming that $E = N^{\mathsf{C}}$, $E = C^{\mathsf{M}}$, $U = L^{\mathsf{M}}$, and $\bot = \top^{\mathsf{M}}$, together with the four "opposite" cases.

Now let $\sqsubseteq$ abbreviate relation subsumption as known from description logics and realized in OWL through `rdfs:subClassOf` (class subsumption) and `rdfs:subPropertyOf` (property subsumption). Given these remarks, we define two further very practical and plausible modal entailments which can be seen as the modal extension of the entailment rules (rdfs9) (for classes) and (rdfs7) (for properties) in RDFS; see [6].

$$\text{(S1)} \quad \mathsf{1}p(\boldsymbol{c}) \wedge p \sqsubseteq q \rightarrow \mathsf{1}q(\boldsymbol{c}) \qquad\qquad \text{(S0)} \quad \mathsf{0}q(\boldsymbol{c}) \wedge p \sqsubseteq q \rightarrow \mathsf{0}p(\boldsymbol{c})$$

Note how the use of p and q switches in the antecedent and the consequent, even though $p \sqsubseteq q$ holds in both cases. Note further that propositional statements π are restricted to the positive case $p(\boldsymbol{c})$ and $q(\boldsymbol{c})$, as their negation in the antecedent will not lead to any valid entailments. Here are four *instantiations* of (S0) and (S1) (remember, $C \in \mathsf{1}$ and $E \in \mathsf{0}$):

$C\mathsf{ViralHepatitisB}(x) \wedge \mathsf{ViralHepatitisB} \sqsubseteq \mathsf{ViralHepatitis} \rightarrow C\mathsf{ViralHepatitis}(x)$
$E\mathsf{Hepatitis}(x) \wedge \mathsf{ViralHepatitis} \sqsubseteq \mathsf{Hepatitis} \rightarrow E\mathsf{ViralHepatitis}(x)$

$C\mathsf{deeplyEnclosedIn}(x, y) \wedge \mathsf{deeplyEnclosedIn} \sqsubseteq \mathsf{containedIn} \rightarrow C\mathsf{containedIn}(x, y)$
$E\mathsf{containedIn}(x, y) \wedge \mathsf{superficiallyLocatedIn} \sqsubseteq \mathsf{containedIn}$
$\rightarrow E\mathsf{superficiallyLocatedIn}(x, y)$

5.3 Extended RDFS & OWL Entailments

In this section, we will consider some of the entailment rules for RDFS [6] and a restricted subset of OWL [18]. Remember that modals only head literals π, neither TBox nor RBox axioms. Concerning the original entailment rules, we will distinguish *four principal cases* to which the extended rules belong (we will only consider the unary and binary case here as used in description logics/OWL):

1. TBox and RBox axiom schemas will not undergo a modal extension;
2. rules get extended in the antecedent;
3. rules take over the modal from the antecedent to the consequent;
4. rules aggregate several modals from the antecedent in the consequent.

We will illustrate the individual cases in the following subsections with examples by using a kind of description logic syntax. Clearly, the set of extended entailments depicted here is *not complete*.

Case-1 Rules: No Modals Entailment rule rdfs11 from [6] deals with class subsumption: $\mathsf{C} \sqsubseteq \mathsf{D} \wedge \mathsf{D} \sqsubseteq \mathsf{E} \rightarrow \mathsf{C} \sqsubseteq \mathsf{E}$. As this is a terminological axiom schema, the rule stays *constant* in the modal domain. Example:

$\mathsf{ViralHepatitisB} \sqsubseteq \mathsf{ViralHepatitis} \wedge \mathsf{ViralHepatitis} \sqsubseteq \mathsf{Hepatitis}$
$\rightarrow \mathsf{ViralHepatitisB} \sqsubseteq \mathsf{Hepatitis}$

Case-2 Rules: Modals on LHS, No or $\top$ Modals on RHS The following original rule rdfs3 from [6] imposes a range restriction on objects of binary ABox relation instances: $\forall \mathsf{P.C} \wedge \mathsf{P(x, y)} \rightarrow \mathsf{C(y)}$.
The extended version (which we call Mrdfs3) needs to address the proposition in the antecedent, but must not change the consequent (even though we always use the $\top$ modality here for typing; see Section 2):

$$(\mathsf{Mrdfs3}) \quad \forall \mathsf{P.C} \wedge \delta \mathsf{P(x, y)} \rightarrow \top \mathsf{C(y)}$$

Example: $\forall \mathsf{suffersFrom.Disease} \wedge L\mathsf{suffersFrom}(x, y) \rightarrow \top \mathsf{Disease}(y)$

Case-3 Rules: Keeping LHS Modals on RHS Inverse properties switch their arguments [18]: $\mathsf{P} \equiv \mathsf{Q}^{-} \wedge \mathsf{P(x, y)} \rightarrow \mathsf{Q(y, x)}$.
The extended version of rdfp8 simply keeps the modal operator:

$$(\mathsf{Mrdfp8}) \quad \mathsf{P} \equiv \mathsf{Q}^{-} \wedge \delta \mathsf{P(x, y)} \rightarrow \delta \mathsf{Q(y, x)}$$

Example: $\mathsf{containedIn} \equiv \mathsf{contains}^{-} \wedge C\mathsf{containedIn}(x, y) \rightarrow C\mathsf{contains}(y, x)$

Case-4 Rules: Aggregating LHS Modals on RHS Now comes the most interesting case of modalized RDFS/OWL entailment rules that offers several possibilities on a varying scale between *skeptical* and *credulous* entailments, depending on the degree of uncertainty, as expressed by the measuring function μ of the modal operator. Consider the original rule $\mathsf{rdfp4}$ from [18] for transitive properties P: $\mathsf{P}^+ \sqsubseteq \mathsf{P} \wedge \mathsf{P(x,y)} \wedge \mathsf{P(y,z)} \rightarrow \mathsf{P(x,z)}$.
How does the modal on the RHS of the extended rule look like, depending on the two LHS modals? There are several possibilities. By operating directly on the *modal hierarchy*, we are allowed to talk about, e.g., the *least upper bound* or the *greatest lower bound* of δ and δ'. When taking the associated *confidence intervals* into account, we might even play with the low and high number of the intervals, say, by applying the *arithmetic mean* or simply by *multiplying* the corresponding numbers.
Let us first consider the general rule from which more specialized versions can be derived, simply by instantiating the combination operator $\odot$:

$$(\mathsf{Mrdfp4}) \quad \mathsf{P}^+ \sqsubseteq \mathsf{P} \wedge \delta\mathsf{P(x,y)} \wedge \delta'\mathsf{P(y,z)} \rightarrow (\delta \odot \delta')\mathsf{P(x,z)}$$

Here is an instantiation of $\mathsf{Mrdfp4}$ dealing with the transitive relation $\mathsf{contains}$ from above: $C\mathsf{contains}(x,y) \wedge L\mathsf{contains}(y,z) \rightarrow (C \odot L)\mathsf{contains}(x,z)$

What is the result of $C \odot L$ here? It depends. Probably both on the application domain and the epistemic commitment one is willing to accept about the "meaning" of gradation words/modal operators. To enforce that $\odot$ is at least both *commutative* and *associative* is probably a good idea, making the sequence of modal clauses order-independent.

5.4 Custom Entailments

Custom entailments are inference rules that are not derived from universal non-modalized RDFS and OWL entailment rules (Section 5.3), but have been formulated to capture the domain knowledge of experts (e.g., physicians). Here is an example. Consider that Hepatitis B is an infectious disease

$$\mathsf{ViralHepatitisB} \sqsubseteq \mathsf{InfectiousDisease} \sqsubseteq \mathsf{Disease}$$

and note that there exist vaccines against it. Assume that the liver l of patient p quite hurts (modal C), but p has been definitely vaccinated (modal $\top$) against Hepatitis B before:

$$C\mathsf{hasPain}(p,l) \wedge \top\mathsf{vaccinatedAgainst}(p, \mathsf{ViralHepatitisB})$$

Given that p received a vaccination, the following custom rule will *not* fire (x and y below are now universally-quantified variables; z an existentially-quantified RHS-only variable):

$$\top\mathsf{Patient}(x) \wedge \top\mathsf{Liver}(y) \wedge C\mathsf{hasPain}(x,y) \wedge U\mathsf{vaccinatedAgainst}(x, \mathsf{ViralHepatitisB})$$
$$\rightarrow N\mathsf{ViralHepatitisB}(z) \wedge N\mathsf{suffersFrom}(x,z)$$

Now assume another person p' that is pretty sure (s)he was never vaccinated:

$$E\mathsf{vaccinatedAgainst}(p', \mathsf{ViralHepatitisB})$$

Given the above custom rule, we are allowed to infer that (h instantiation of z)

$N\mathsf{ViralHepatitisB}(h) \wedge N\mathsf{suffersFrom}(p', h)$

The subclass axiom from above thus assigns

$N\mathsf{InfectiousDisease}(h)$

so that we can query for patients for whom an infectious disease is *not unlikely*, in order to initiate appropriate methods (e.g., further medical investigations).

6 Related Approaches and Remarks

It is worth noting to state that this paper is interested in the representation of and reasoning with *uncertain assertional* knowledge, and neither in dealing with *vagueness* found in natural language (*very small*), nor in handling *defaults* and *exceptions* in *terminological* knowledge (*penguins can't fly*).

To the best of our knowledge, the modal logic presented in this paper uses for the first time modal operators for expressing the degree of (un)certainty of propositions. These modal operators are interpreted in the model theory through confidence intervals, by using a measure function μ. From a model point of view, our modal operators are related to *counting modalities* $\Diamond^{\geq k}$ [4, 1]—however, we do *not* require a *fixed* number $k \in \mathbb{N}$ of reachable successor states (*absolute* frequency), but instead *divide* the number of worlds v reached through label $\delta \in \triangle$ by the number of all reachable worlds, given current state w, yielding $0 \leq p \leq 1$. This fraction then is further constrained by requiring $p \in \mu(\delta)$ (*relative* frequency), as defined in case 5. of the satisfaction relation in Sections 4.2 and 4.3.

As [20] precisely put it: "*... what axioms and rules must be added to the propositional calculus to create a usable system of modal logic is a matter of philosophical opinion, often driven by the theorems one wishes to prove ...*". Clearly, the logic Λ is *no* exception and its design is driven by commonsense knowledge and plausible inferences, we try to capture.

Our modal logic can be regarded as an instance of the *normal* modal logic $\mathbf{K} := (N) + (K)$ when identifying the basic modal operator $\Box$ with the modal $\top$ (and *only* with $\top$) and by enforcing the *well-behaved* frame condition from Section 4.3. Given $\Box \equiv \top$, Λ then includes the *necessitation rule* (N) $p \rightarrow \top p$ and the *distribution axiom* (K) $\top(p \rightarrow q) \rightarrow (\top p \rightarrow \top q)$ where p, q being special theorems in Λ, viz., positive and negative propositional letters.

(N) can be seen as a special case of (L), the *Lift* modal entailment (left-to-right direction) from Section 5.1. (K) can be proven in Λ by choosing $\top \in \underline{1}$ in simplification rule *8b* (Section 4.1) and by instantiating (G), the *Generalize* modal entailment (Section 5.1), together with the application of the tautology $(p \rightarrow q) \Leftrightarrow (\neg p \vee q)$:

$$\frac{\frac{\frac{\frac{\top(p \rightarrow q) \rightarrow (\top p \rightarrow \top q)}{\top(\neg p \vee q) \rightarrow (\neg \top p \vee \top q)}}{(\top \neg p \vee \top q) \rightarrow (\neg \top p \vee \top q)}}{\top \neg p \rightarrow \neg \top p}}{\underline{\bot p \rightarrow \top^{\mathsf{C}} p}}$$

The final simplification at which we arrive is valid, since $\bot \preceq \top^{\mathsf{C}}$:

$$\mu(\bot) = [0,0] \subseteq [0,1) = \mu(\top^{\mathsf{C}})$$

Again, through (L) (right-to-left direction), Λ also incorporates the *reflexivity axiom* (T) $\top p \rightarrow p$ making Λ (at least) an instance of the system **T**. However, this investigation is in a certain sense *useless* as it does *not* address the other modals: almost always, neither (N), (K), nor (T) hold for modals from $\triangle$. Thus, we can *not* view Λ as an instance of a *poly-modal* logic.

Several approaches to representing and reasoning with uncertainty have been investigated in *Artificial Intelligence* (see [14, 5] for two comprehensive overviews). Very less so has been researched in the *Description Logic* community, and little or nothing of this research has find its way into implemented systems. [7] and [8] consider *uncertainty* in $\mathcal{ALC}$ concept hierarchies, plus concept typing of individuals (unary relations) in different ways (probability values vs. intervals; conditional probabilities in TBox vs. ABox). They do not address uncertain binary (or even n-ary) relations. [19] investigates *vagueness* in $\mathcal{ALC}$ concept descriptions to address statements, such as *the patient's temperature is high*, but also for determining membership degree (*38.5 ℃*). This is achieved through *membership manipulators* which are functions, returning a truth value between 0 and 1, thus deviating from a two-valued logic. [17] defines a *fuzzy* extension of $\mathcal{ALC}$, based on Zadeh's *fuzzy logic*. As in [19], the truth value of an assertion is replaced by a membership value from $[0,1]$. $\mathcal{ALC}$ assertions α in [17] are made fuzzy by writing, e.g., $\langle \alpha \geq n \rangle$, thus taking a single truth value from $[0,1]$. An even more expressive description logic, Fuzzy OWL, based on OWL DL, is investigated in [16].

Our work might be viewed as a modalized version of a restricted fragment of *Subjective Logic* [9, 10], a probabilistic logic that can be seen as an extension of Dempster-Shafer belief theory. Subjective Logic addresses subjective believes by requiring numerical values for *believe* b, *disbelieve* d, and *uncertainty* u, called (subjective) opinions. For each proposition, it is required that $b + d + u = 1$. The translation from modals δ to $\langle b, d, u \rangle$ is determined by the length of the confidence interval $\mu(\delta) = [l, h]$ and its starting/ending numbers, viz., $u := h - l$, $b := l$, and $d := 1 - h$.

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